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Research Article

On Propagation of Sphericity of Real Analytic Hypersurfaces across Levi Degenerate Loci

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A connected real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ whose Levi form is nondegenerate in at least one point—hence at every point of some Zariski-dense open subset—is locally biholomorphic to the model Heisenberg quadric pseudosphere of signature $(k, n-k)$ in one point if and only if, at every other Levi nondegenerate point, it is also locally biholomorphic to some Heisenberg pseudosphere, possibly having a different signature $(l, n-l)$. Up to signature, pseudosphericity then jumps across the Levi degenerate locus and in particular across the nonminimal locus, if there exists any.

1. Introduction

The goal of this paper is to provide the complete details of an alternative direct proof of a recent theorem due to Kossovskiy and Shafikov ([1]) which relies on the explicit zero-curvature equations obtained in [2, 3], following the lines of a clever suggestion of Beloshapka. In fact, the proof we give here freely brings a more general statement.

Let $M \subset \mathbb{C}^{n+1}$ be a connected real analytic hypersurface with $n \geqslant 1$. One says that M is $(k, n-k)$ *pseudospherical* at one of its points p if it is locally near p biholomorphic to some Heisenberg $(k, n-k)$-pseudosphere having, in coordinates $(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1}$, the model quadric equation:

$$w = \overline{w} + 2i\left(-z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n\right), \tag{1}$$

for some integer k with $0 \leqslant k \leqslant n-k$; when $n = 1$, one simply says that M is *spherical*. One calls $(k, n-k)$ the *signature*.

It is known (*see* [1] and the references therein) that a connected real analytic hypersurface of $\mathbb{C}^{n+1}$ which is Levi nondegenerate at every point is $(k, n-k)$-pseudospherical at one point if and only if it is $(k, n-k)$-pseudospherical at every point. More generally, we establish that propagation of $(k, n-k)$ pseudosphericality also holds in presence of Levi degenerate points of arbitrary kind.

Theorem 1. *Let $M \subset \mathbb{C}^{n+1}$ be a connected real analytic geometrically smooth hypersurface which is Levi nondegenerate in at least one point (hence in some nonempty open subset). Then one has the following:*

(a) *The set of Levi nondegenerate points of M is a Zariski open subset of M in the sense that there exists a certain proper—that is, having dimension $\leqslant \dim M - 1 = 2n$ —locally closed global real analytic subset $\Sigma_{\mathsf{LD}} \subset M$ locating exactly the Levi degenerate points of M:*

$$p \in M \setminus \Sigma_{\mathsf{LD}} \Longleftrightarrow \text{Levi form of } M \text{ at } p \text{ is nondegenerate.} \tag{2}$$

(b) *If M is locally biholomorphic, in a neighborhood of one of its points $p \in M$, to some Heisenberg $(k, n-k)$-pseudosphere having equation,*

$$w = \overline{w} + 2i\left(-z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n\right), \tag{3}$$

for some integer k with $0 \leqslant k \leqslant n-k$ (so that $p \in M \setminus \Sigma_{\mathsf{LD}}$ necessarily is a Levi nondegenerate point of M too), then locally at every other Levi nondegenerate point $q \in M \setminus \Sigma_{\mathsf{LD}}$, the hypersurface M is also locally biholomorphic to some Heisenberg $(l, n-l)$-pseudosphere with possibly $l \neq k$.

Surprisingly, Example 6.2 in [1] shows that $l \neq k$ may occur, in the case of a *nonminimal* hypersurface of $\mathbb{C}^{n+1}$ with $n \geqslant 2$, a local example for which the Levi degenerate locus Σ_{LD} consists precisely of a complex n-dimensional hypersurface contained in M.

In [1], a similar theorem was proved, assuming that $\Sigma_{\mathsf{LD}} = S$ consists of a complex hypersurface $S \subset M$ (nonminimal case), while $M \setminus S$ consists only of Levi nondegenerate points.

In the previous theorem, Σ_{LD} can be arbitrary.

2. Proof in $\mathbb{C}^2$

Let $M \subset \mathbb{C}^2$ be a connected real analytic hypersurface. Pick a point

$$p \in M, \tag{4}$$

and choose some affine coordinates centered at p:

$$(z, w) = (x + iy, u + iv), \tag{5}$$

satisfying

$$T_0 M = \{\operatorname{Im} w = 0\}, \tag{6}$$

so that the implicit function theorem represents M as

$$u = \varphi(x, y, v), \tag{7}$$

in terms of some *graphing function* φ which is expandable in convergent Taylor series in some (possibly small) open bidisc:

$$\square^2_{\rho_0} := \left\{(z, w) \in \mathbb{C}^2 \colon \ |z| < \rho_0, \ |w| < \rho_0\right\}, \tag{8}$$

with of course $\rho_0 > 0$.

Classically, writing

$$\frac{w + \overline{w}}{2} = \varphi\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}, \frac{w - \overline{w}}{2i}\right) \tag{9}$$

and using the analytic implicit function theorem, one solves w in terms of z, $\overline{z}$, and $\overline{w}$ getting a representation of M as

$$w = \Theta(z, \overline{z}, \overline{w}); \tag{10}$$

recall that implicitly when one does this, one must consider $(z, w, \overline{z}, \overline{w})$ as 4 independent complex variables, which amounts to *complexify* them, namely, to introduce the *complexified variables*:

$$(z, w, \underline{z}, \underline{w}) \in \mathbb{C}^4; \tag{11}$$

in what follows, we will work with $(z, w, \overline{z}, \overline{w})$-variables, keeping in mind that they can be replaced by $(z, w, \underline{z}, \underline{w})$ *since all objects are convergent Taylor series*; so here, $\Theta(z, \underline{z}, \underline{w})$ is a convergent Taylor series of $(z, \underline{z}, \underline{w})$ for $|z| < \rho_0$, $|\underline{z}| < \rho_0$, and $|\underline{w}| < \rho_0$, after shrinking $\rho_0 > 0$ if necessary.

Moreover, since

$$0 = \varphi(0) = \varphi_x(0) = \varphi_y(0) = \varphi_u(0), \tag{12}$$

one has

$$\Theta = \overline{w} + O(2). \tag{13}$$

Now, it is known ([4, 5])—or it could be taken here as a definition—that M is *Levi nondegenerate* at $0 \in M$ when the local holomorphic map

$$\begin{gathered} \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \\ (\overline{z}, \overline{w}) \longmapsto \left(\Theta(0, \overline{z}, \overline{w}), \Theta_z(0, \overline{z}, \overline{w})\right) \end{gathered} \tag{14}$$

is of rank 2 at $(\overline{z}, \overline{w}) = (0, 0)$; of course, one would better think in terms of $(\underline{z}, \underline{w})$-variables here.

More generally, M is *Levi nondegenerate* at an arbitrary point close to the origin:

$$\left(z_p, w_p\right) \in M \cap \square^2_{\rho_0} \tag{15}$$

when the map

$$\begin{gathered} \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \\ (\overline{z}, \overline{w}) \longmapsto \left(\Theta\left(z_p, \overline{z}, \overline{w}\right), \Theta_z\left(z_p, \overline{z}, \overline{w}\right)\right) \end{gathered} \tag{16}$$

is of rank 2 at $(\overline{z}_p, \overline{w}_p)$, which precisely means the nonvanishing of the Jacobian determinant:

$$0 \neq \det\begin{pmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{pmatrix} = \Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}} \tag{17}$$

at

$$(z, \overline{z}, \overline{w}) = \left(z_p, \overline{z}_p, \overline{w}_p\right). \tag{18}$$

One may either show-check that such a definition regives the standard definition of Levi nondegeneracy (*cf.* [4, 5]) or prove directly that, as it stands, it really is independent of the choice of coordinates ([6]).

Although we could then spend time to reprove it properly, the following fact—here admitted—is well known.

Proposition 2. *If a connected real analytic hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is Levi nondegenerate in at least one point, then the set of Levi degenerate points of M is a proper real analytic subset:*

$$\Sigma_{\mathsf{LD}} \subsetneq M. \tag{19}$$

Suppose to begin with for $M^3 \subset \mathbb{C}^2$ that

$$0 \notin \Sigma_{\mathsf{LD}}. \tag{20}$$

Since the above map is of rank 2 at $(\overline{z}, \overline{w}) = (0, 0)$, one can solve, following [2], the two equations:

$$\begin{aligned} w(z) &= \Theta(z, \overline{z}, \overline{w}), \\ w_z(z) &= \Theta_z(z, \overline{z}, \overline{w}), \end{aligned} \tag{21}$$

for the two variables $(\overline{z}, \overline{w})$, and then insert the latter in

$$w_{zz}(z) = \Theta_{zz}(z, \overline{z}, \overline{w}), \tag{22}$$

to get a complex second-order ordinary differential equation:

$$w_{zz}(z) = \Phi\left(z, w(z), w_z(z)\right). \tag{23}$$

One should notice that the possibility of solving $(\overline{z}, \overline{w})$ is expressed by the nonvanishing of *exactly and precisely the same* Jacobian determinant as the one expressing Levi nondegeneracy.

In [2], one deduces from an explicitly known condition on Φ for this second-order equation $w_{zz} = \Phi(z, w, w_z)$ to be pointwise equivalent to the free particle Newtonian equation

$$w'_{z'z'} = 0 \tag{24}$$

that a real analytic hypersurface $M \subset \mathbb{C}^2$ which is Levi nondegenerate at $0 \in M$ as above is *spherical* in the sense—recall the definition—that it is locally biholomorphic to

$$w' = \overline{w}' + 2iz'\overline{z}', \tag{25}$$

if and only if its complex graphing function Θ satisfies an explicit (not completely developed) equation which we now present.

Introduce the expression

$$\begin{aligned}
&\mathsf{AJ}^4(\Theta) \\
&:= \frac{1}{\left[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}\right]^3} \left\{\Theta_{zz\overline{z}\,\overline{z}} \left(\Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix}\right)\right. \\
&- 2\Theta_{zz\overline{z}\,\overline{w}} \left(\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix}\right) \\
&+ \Theta_{zz\overline{w}\,\overline{w}} \left(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}} \end{vmatrix}\right) \\
&+ \Theta_{zz\overline{z}} \left(\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{w}\,\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{w}\,\overline{w}} \end{vmatrix} - 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\,\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\,\overline{w}} \end{vmatrix}\right. \\
&\left. + \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{w}} & \Theta_{\overline{z}\,\overline{z}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\,\overline{z}} \end{vmatrix}\right) \\
&+ \Theta_{zz\overline{w}} \left(-\Theta_{\overline{z}}\Theta_{\overline{z}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{w}\,\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}\,\overline{w}} \end{vmatrix} + 2\Theta_{\overline{z}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\,\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\,\overline{w}} \end{vmatrix}\right. \\
&\left.\left. - \Theta_{\overline{w}}\Theta_{\overline{w}} \begin{vmatrix} \Theta_{\overline{z}} & \Theta_{\overline{z}\,\overline{z}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\,\overline{z}} \end{vmatrix}\right)\right\},
\end{aligned} \tag{26}$$

noticing that its denominator

$$\left[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}\right]^3 \tag{27}$$

does not vanish at the origin since $0 \in M$ was assumed (temporarily) to be a Levi nondegenerate point. Introduce also the vector field derivation

$$\mathscr{D} := \frac{-\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}} \frac{\partial}{\partial \overline{z}} + \frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}} \frac{\partial}{\partial \overline{w}}. \tag{28}$$

Then the main and unique theorem of [2] states that M is spherical at 0 if and only if

$$0 \equiv \mathscr{D}\left(\mathscr{D}\left(\mathsf{AJ}^4(\Theta)\right)\right), \tag{29}$$

identically in $\mathbb{C}\{z, \overline{z}, \overline{w}\}$.

Unfortunately, it is essentially impossible to print in a published article what one obtains after a full expansion of these two derivations; other instances of this phenomenon appear in [7, 8].

Nevertheless, by thinking a bit, one convinces oneself that, after full expansion and reduction to a common denominator, one obtains a kind of expression that we will denote in summarized form as

$$\frac{\mathsf{polynomial}\left(\left(\Theta_{z^j\overline{z}^k\overline{w}^l}\right)_{1\leqslant j+k+l\leqslant 6}\right)}{\left[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}\right]^7}, \tag{30}$$

and hence instantly, sphericity of M is characterized by

$$0 \equiv \mathsf{polynomial}\left(\left(\Theta_{z^j\overline{z}^k\overline{w}^l}(z, \overline{z}, \overline{w})\right)_{1\leqslant j+k+l\leqslant 6}\right). \tag{31}$$

One notices that the complex graphing function Θ is differentiated always at least once.

Interpretation. Then the true thing is that, after erasing the Levi determinant lying at denominator place, if this explicit equation vanishes in some very small neighborhood of some point

$$\left(z_p, \overline{z}_p, \overline{w}_p\right) \in \square^3_{\rho_0} \tag{32}$$

of the polydisc of convergence of Θ, then by the uniqueness principle for analytic functions, the concerned polynomial numerator

$$\mathsf{polynomial}\left(\left(\Theta_{z^j\overline{z}^k\overline{w}^l}(z, \overline{z}, \overline{w})\right)_{1\leqslant j+k+l\leqslant 6}\right) \tag{33}$$

immediately vanishes identically all over $\square^3_{\rho_0}$, so that sphericity at one point should freely propagate to all other Levi nondegenerate points $q \in M \cap \square^2_{\rho_0}$.

Before providing rigorous details to explain the latter assertion, a further comment is in order.

The explicit sphericity formula brings the information that denominator places are occupied by nondegeneracy conditions, so that division is allowed only at points where these conditions are satisfied, but numerators happen to be polynomial, a computational fact which hence enables one to jump across degenerate points through the "bridge-numerator" from one nondegenerate point to another nondegenerate point.

Now, the local version of Theorem 1 is as follows. Notice that from now on one does not assume anymore that the origin $0 \in M \setminus \Sigma_{\mathsf{LD}}$ is a Levi nondegenerate point.

Proposition 3. *With $M^3 \subset \mathbb{C}^2$, $\square^2_{\rho_0}$, $(z, \overline{z}, \overline{w})$, φ, and Θ as above, assuming that the real analytic subset Σ_{LD} of Levi* *degenerate points is proper, if M is spherical at one Levi nondegenerate point*

$$p \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \square^2_{\rho_0}, \tag{34}$$

then M is also spherical at every other Levi nondegenerate point:

$$q \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \square^2_{\rho_0}. \tag{35}$$

Proof. Take a (possibly much) smaller bidisc

$$p + \square^2_{\rho'} \subset\subset \square^2_{\rho_0}, \tag{36}$$

with $0 < \rho' \ll \rho_0$ to be chosen below, and center new coordinates at

$$p = (z_p, w_p); \tag{37}$$

that is to say, introduce the new translated coordinates:

$$\begin{aligned} z' &:= z - z_p, \\ w' &:= w - w_p. \end{aligned} \tag{38}$$

The graphed complex equation

$$w = \Theta(z, \overline{z}, \overline{w}) \tag{39}$$

then becomes

$$w' + w_p = \Theta\left(z' + z_p, \overline{z}' + \overline{z}_p, \overline{w}' + \overline{w}'_p\right). \tag{40}$$

Of course, the fact that $p \in M$ reads

$$w_p = \Theta\left(z_p, \overline{z}_p, \overline{w}_p\right), \tag{41}$$

and hence, in the new coordinates (z', w') centered at p, the equation of M becomes

$$\begin{aligned} w' &= \Theta\left(z' + z_p, \overline{z}' + \overline{z}_p, \overline{w}' + \overline{w}_p\right) - \Theta\left(z_p, \overline{z}_p, \overline{w}_p\right) \\ &=: \Theta'\left(z', \overline{z}', \overline{w}'\right), \end{aligned} \tag{42}$$

in terms of a new graphing function Θ' that visibly satisfies

$$\Theta'\left(0', 0', 0'\right) = 0. \tag{43}$$

Observation 4. For any integers

$$(j, k, l) \in \mathbb{N}^3, \tag{44}$$

with

$$j + k + l \geqslant 1, \tag{45}$$

one has

$$\Theta'_{z'^j \overline{z}'^k \overline{w}'^l}\left(0', 0', 0'\right) = \Theta_{z^j \overline{z}^k \overline{w}^l}\left(z_p, \overline{z}_p, \overline{w}_p\right). \tag{46}$$

Proof. Indeed, the constant $-\Theta(z_p, \overline{z}_p, \overline{w}_p)$ disappears after just a single differentiation. This completes the proof.

Now, the Levi nondegeneracy of M at p which reads according to what precedes as

$$0 \neq \left[\Theta_{\overline{z}} \Theta_{z\overline{w}} - \Theta_{\overline{w}} \Theta_{z\overline{z}}\right]\left(z_p, \overline{z}_p, \overline{w}_p\right) \tag{47}$$

reads in the new coordinates as (remember that inequation (17) is independent of coordinates)

$$0 \neq \left[\Theta'_{\overline{z}'} \Theta'_{z'\overline{w}'} - \Theta'_{\overline{w}'} \Theta'_{z'\overline{z}'}\right]\left(0', 0', 0'\right), \tag{48}$$

which means as we know Levi nondegeneracy at $(z', \overline{z}', \overline{w}') = (0', 0', 0')$.

Precisely because in [2] one needs only this condition to hold in order to associate as was explained above a second-order complex ordinary differential equation

$$w'_{z'z'} = \Phi'\left(z', w'\left(z'\right), w'_{z'}\left(z'\right)\right) \tag{49}$$

for some possibly very small

$$\begin{aligned} \left|z'\right| &< \rho', \\ \left|w'\right| &< \rho', \end{aligned} \tag{50}$$

this is where one has to choose ρ' with $0 < \rho' \ll \rho_0$, the possible presence of rather close Levi degenerate points being a constraint in the needed application(s) of the implicit function theorem, one has the impression that one can in principle only determine whether M is spherical restrictively in such a very narrow neighborhood $\square^2_{\rho'}$ of p in $\mathbb{C}^2$, when one applies the main result of [2].

But looking just at the numerator of the equation which expresses that M is spherical at p in the coordinates (z', w')

$$0 \equiv \frac{\overbrace{\text{polynomial}}^{\text{same universal expression}}\left(\left(\Theta'_{z'^j \overline{z}'^k \overline{w}'^l}\right)_{1 \leqslant j+k+l \leqslant 6}\right)}{\underbrace{\left[\Theta'_{\overline{z}'} \Theta'_{z'\overline{w}'} - \Theta'_{\overline{w}'} \Theta'_{z'\overline{z}'}\right]^7}_{\text{nonvanishing at } (0', 0', 0')}}, \tag{51}$$

if one takes into account the above observation, one readily realizes that

$$\begin{aligned} &\text{polynomial}\left(\left(\Theta'_{z'^j \underline{z}'^k \underline{w}'^l}\left(z', \underline{z}', \underline{w}'\right)\right)_{1 \leqslant j+k+l \leqslant 6}\right) \\ &\quad = \text{polynomial}\left(\left(\Theta_{z^j \underline{z}^k \underline{w}^l}\left(z, \underline{z}, \underline{w}\right)\right)_{1 \leqslant j+k+l \leqslant 6}\right), \end{aligned} \tag{52}$$

so that the identical vanishing of the left-hand side for

$$\begin{aligned} \left|z'\right| &< \rho' \ll \rho_0, \\ \left|\underline{z}'\right| &< \rho' \ll \rho_0, \\ \left|\underline{w}'\right| &< \rho' \ll \rho_0 \end{aligned} \tag{53}$$

means the identical vanishing of the right-hand side for

$$\begin{aligned}\left|z-z_p\right| &< \rho' \ll \rho_0,\\ \left|\underline{z}-\overline{z}_p\right| &< \rho' \ll \rho_0,\\ \left|\underline{w}-\overline{w}_p\right| &< \rho' \ll \rho_0,\end{aligned} \tag{54}$$

which lastly yields *thanks to the uniqueness principle enjoyed by analytic functions* the identical vanishing of the original numerator *in the whole initial domain of convergence*:

$$0 \equiv \mathsf{polynomial}\left(\left(\Theta_{z^j\underline{z}^k\underline{w}^l}\left(z,\underline{z},\underline{w}\right)\right)_{1\leqslant j+k+l\leqslant 6}\right) \qquad \left(|z|<\rho_0,\ |\underline{z}|<\rho_0,\ |\underline{w}|<\rho_0\right). \tag{55}$$

Take now any other Levi nondegenerate point:

$$q \in M \cap \square^2_{\rho_0}. \tag{56}$$

The goal is to prove that M is also spherical at q. Center similarly new coordinates at $q=(z_q,w_q)$:

$$\begin{aligned}z'' &:= z - z_q,\\ w'' &:= w - w_q.\end{aligned} \tag{57}$$

Introduce the new graphed equations:

$$\begin{aligned}w'' &= \Theta\left(z''+z_q,\overline{z}''+\overline{z}_q,\overline{w}''+\overline{w}_q\right)-\Theta\left(z_q,\overline{z}_q,\overline{w}_q\right)\\ &=: \Theta''\left(z'',\overline{z}'',\overline{w}''\right).\end{aligned} \tag{58}$$

At such a point, since the Levi determinant is nonvanishing, one can for completeness construct the associated second-order complex ordinary differential equations

$$w''_{z''z''}\left(z''\right) = \Phi''\left(z'',w''\left(z''\right),w''_{z''}\left(z''\right)\right) \tag{59}$$

or question directly whether local sphericity holds near q by plainly applying the main theorem of [2]; namely, question whether the following equation holds:

$$0 \overset{?}{\equiv} \frac{\overbrace{\mathsf{polynomial}}^{\text{again same universal expression}}\left(\left(\Theta''_{z''^j\underline{z}''^k\underline{w}''^l}\right)_{1\leqslant j+k+l\leqslant 6}\right)}{\underbrace{\left[\Theta''_{\underline{z}''}\Theta''_{z''\underline{w}''}-\Theta''_{\underline{w}''}\Theta''_{z'\underline{z}''}\right]^7}_{\text{nonvanishing at }(0'',0'',0'')}}. \tag{60}$$

But then by exactly the same application of the above observation, we know that this last numerator satisfies

$$\begin{aligned}&\mathsf{polynomial}\left(\left(\Theta''_{z''^j\underline{z}''^k\underline{w}''^l}\left(z'',\underline{z}'',\underline{w}''\right)\right)_{1\leqslant j+k+l\leqslant 6}\right)\\ &\quad = \mathsf{polynomial}\left(\left(\Theta_{z^j\underline{z}^k\underline{w}^l}\left(z,\underline{z},\underline{w}\right)\right)_{1\leqslant j+k+l\leqslant 6}\right),\end{aligned} \tag{61}$$

when

$$\begin{aligned}\left|z''\right| &< \rho'' \ll \rho_0,\\ \left|\underline{z}''\right| &< \rho'' \ll \rho_0,\\ \left|\underline{w}'\right| &< \rho'' \ll \rho_0,\\ \left|z-z_q\right| &< \rho'' \ll \rho_0,\\ \left|\underline{z}-\overline{z}_q\right| &< \rho'' \ll \rho_0,\\ \left|\underline{w}-\overline{w}_q\right| &< \rho'' \ll \rho_0,\end{aligned} \tag{62}$$

and since we already know that the latter right-hand side vanishes, according to the last boxed equation, we conclude that M is indeed spherical at q. □

To finish the proof of Theorem 1 in the case $n = 1$, it remains only to *globalize* this local propagation of sphericity.

Assume therefore that $M \subset \mathbb{C}^2$ is a connected real analytic smooth hypersurface, that is $\Sigma_{\mathrm{LD}} \neq M$, so that the set $M \setminus \Sigma_{\mathrm{LD}}$ of Levi nondegenerate points is Zariski open, everywhere dense in M. Assume that there exists a *spherical point* $p \in M \setminus \Sigma_{\mathrm{LD}}$ (abbreviation for "an open neighborhood of p in M is locally biholomorphic to the unit 3-sphere $S^3 \subset \mathbb{C}^2$"). Take any other point $q \in M \setminus \Sigma_{\mathrm{LD}}$.

By connectedness of M, there exists a finite (possibly large) number $\nu \geqslant 1$ of points

$$p = p_0, p_1, \dots, p_{\nu-1}, p_\nu = q, \tag{63}$$

and there exist small radii $\rho_0,\dots,\rho_\nu > 0$ such that, in the open polydiscs,

$$\Delta_\kappa := p_\kappa + \square^2_{\rho_\kappa} \quad (0 \leqslant \kappa \leqslant \nu), \tag{64}$$

the local hypersurface $M \cap \Delta_\kappa$ is connected and graphed either as

$$w - w_{p_\kappa} = \Theta^\kappa\left(z - z_{p_\kappa}, \overline{z - z_{p_\kappa}}, \overline{w - w_{p_\kappa}}\right) \tag{65}$$

or as

$$z - z_{p_\kappa} = \Xi^\kappa\left(w - w_{p_\kappa}, \overline{w - w_{p_\kappa}}, \overline{z - z_{p_\kappa}}\right), \tag{66}$$

and such that $[M \cap \Delta_{\kappa-1}] \cap [M \cap \Delta_\kappa]$ is *open nonempty* in M.

Since $\Sigma_{\mathrm{LD}} \subsetneqq M$ is locally stratified by a finite union of real submanifolds of dimension $< 3 = \dim_{\mathbb{R}} M$, this insures that

$$\begin{aligned}&\emptyset \neq \left(M \setminus \Sigma_{\mathrm{LD}}\right) \cap \Delta_\kappa \quad (\forall 0 \leqslant \kappa \leqslant \nu),\\ &\emptyset \neq \left[\left(M \setminus \Sigma_{\mathrm{LD}}\right) \cap \Delta_{\kappa-1}\right] \cap \left[\left(M \setminus \Sigma_{\mathrm{LD}}\right) \cap \Delta_\kappa\right] \quad (\forall 1 \leqslant \kappa \leqslant \nu).\end{aligned} \tag{67}$$

Remember that the main Proposition 3 that precedes has shown that, for every $0 \leqslant \kappa \leqslant \nu$,

$$\begin{aligned}&\exists r \in \left(M \setminus \Sigma_{\mathrm{LD}}\right) \cap \Delta_\kappa \text{ spherical} \Longrightarrow\\ &\text{all } r \in \left(M \setminus \Sigma_{\mathrm{LD}}\right) \cap \Delta_\kappa \text{ are spherical.}\end{aligned} \tag{68}$$

Then an immediate induction yields propagation from

$$
\begin{aligned}
&p = p_0 \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_0 \text{ is spherical} \Longrightarrow \\
&\text{all } r \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_0 \cap \Delta_1 \text{ are spherical} \Longrightarrow \\
&\text{all } r \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_1 \text{ are spherical} \\
&\vdots \Longrightarrow \\
&\text{all } r \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_{\nu-1} \cap \Delta_\nu \text{ are spherical} \Longrightarrow \\
&\text{all } r \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_\nu \text{ are spherical,}
\end{aligned}
\tag{69}
$$

up to $q \in (M \setminus \Sigma_{\mathrm{LD}}) \cap \Delta_\nu$ thereby shown to be spherical too.

3. Proof in $\mathbb{C}^{n+1}$ $(n \geqslant 2)$

We briefly summarize the quite similar arguments, relying upon [3], which is unpublished and several times rejected, although useful here.

The Levi determinant becomes

$$
\Delta := \begin{vmatrix} \Theta_{\overline{z}_1} & \cdots & \Theta_{\overline{z}_n} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \cdots & \Theta_{z_1\overline{z}_n} & \Theta_{z_1\overline{w}} \\ \cdots & \cdots & \cdots & \cdots \\ \Theta_{z_n\overline{z}_1} & \cdots & \Theta_{z_n\overline{z}_n} & \Theta_{z_n\overline{w}} \end{vmatrix}. \tag{70}
$$

It is nonzero at one point

$$
p = \left(z_{1p}, \ldots, z_{np}, w_p\right) \in M \tag{71}
$$

if and only if M is Levi nondegenerate at p and also if and only if one can associate with M a completely integrable system of second-order partial differential equations:

$$
w_{z_{k_1} z_{k_2}}(z) = \Phi_{k_1,k_2}\left(z, w(z), w_{z_1}(z), \ldots, w_{z_n}(z)\right) \qquad \left(1 \leqslant k_1, k_2 \leqslant n\right). \tag{72}
$$

Hachtroudi ([9]) established that such a system is pointwise equivalent to

$$
w'_{z'_{k_1} z'_{k_2}}\left(z'\right) = 0 \quad \left(1 \leqslant k_1, k_2 \leqslant n\right) \tag{73}
$$

if and only if

$$
\begin{aligned}
0 \equiv {} & \frac{\partial^2 \Phi_{k_1,k_2}}{\partial w_{z_{\ell_1}} w_{z_{\ell_2}}} - \frac{1}{n+2} \sum_{\ell_3=1}^{n} \left(\delta_{k_1,\ell_1} \frac{\partial^2 \Phi_{\ell_3,k_2}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_2}}} \right. \\
& + \delta_{k_1,\ell_2} \frac{\partial^2 \Phi_{\ell_3,k_2}}{\partial w_{z_{\ell_1}} \partial w_{z_{\ell_3}}} + \delta_{k_2,\ell_1} \frac{\partial^2 \Phi_{k_1,\ell_3}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_2}}} \\
& \left. + \delta_{k_2,\ell_2} \frac{\partial^2 \Phi_{k_1,\ell_3}}{\partial w_{z_{\ell_1}} \partial w_{z_{\ell_3}}} \right) + \frac{1}{(n+1)(n+2)} \left[\delta_{k_1,\ell_1} \delta_{k_2,\ell_2} \right. \\
& \left. + \delta_{k_2,\ell_1} \delta_{k_1,\ell_2} \right] \sum_{\ell_3=1}^{n} \sum_{\ell_4=1}^{n} \frac{\partial^2 \Phi_{\ell_3,\ell_4}}{\partial w_{z_{\ell_3}} \partial w_{z_{\ell_4}}} \\
& \left(1 \leqslant k_1, k_2 \leqslant n\right) \ \left(1 \leqslant \ell_1, \ell_2 \leqslant n\right).
\end{aligned}
\tag{74}
$$

When one does apply Hachtroudi's results to CR geometry (instead of Chern-Moser's results, which is up to now not sufficiently explicit to be applied), the signature of the Levi forms disappears for the following reason.

The infinite-dimensional local Lie (pseudo)group of biholomorphic transformations

$$
\begin{aligned}
\left(z_1, \ldots, z_n, w\right) &\longmapsto \left(z'_1, \ldots, z'_n, w'\right) \\
&= \left(z'_1\left(z_\bullet, w\right), \ldots, z'_n\left(z_\bullet, w\right), w'\left(z_\bullet, w\right)\right)
\end{aligned}
\tag{75}
$$

acts simultaneously on $(z_\bullet, w)$-variables and on $(\overline{z}_\bullet, \overline{w})$-variables as

$$
\begin{aligned}
\left(\overline{z}_1, \ldots, \overline{z}_n, \overline{w}\right) &\longmapsto \left(\overline{z}'_1, \ldots, \overline{z}'_n, \overline{w}'\right) \\
&= \left(\overline{z}'_1\left(\overline{z}_\bullet, \overline{w}\right), \ldots, \overline{z}'_n\left(\overline{z}_\bullet, \overline{w}\right), \overline{w}'\left(\overline{z}_\bullet, \overline{w}\right)\right).
\end{aligned}
\tag{76}
$$

But when one passes to the extrinsic complexification, one replaces $(\overline{z}, \ldots, \overline{z}_n, \overline{w})$-variables by new independent variables:

$$
\left(\underline{z}_1, \ldots, \underline{z}_n, \underline{w}\right), \tag{77}
$$

considered as the *constants* of integration for the system of partial differential equations. Hence the local Lie (pseudo)group considered by Hachtroudi becomes enlarged as the group of transformations:

$$
\begin{aligned}
\left(z_\bullet, w, \underline{z}_\bullet, \underline{w}\right) \longmapsto \big(& \text{holomorphic map}\left(z_\bullet, w\right), \\
& \text{other holomorphic map}\left(\underline{z}_\bullet, \underline{w}\right)\big)
\end{aligned}
\tag{78}
$$

in which the transformations on the "constant-of-integration" variables $(\underline{z}_1, \ldots, \underline{z}_n, \underline{w})$ become completely discoupled from the group of transformations on the true variables $(z_1, \ldots, z_n, w)$. By definition (*cf.* the explanations in [2]), transformations on differential equations, when viewed in the space of solutions, are always of this general form.

It is then clear that all *complexified* Heisenberg $(k, n-k)$ pseudospheres

$$
\begin{aligned}
w &= \underline{w} \\
&\quad + 2i\left(-z_1\underline{z}_1 - \cdots - z_k\underline{z}_k + z_{k+1}\underline{z}_{k+1} + \cdots + z_n\underline{z}_n\right)
\end{aligned}
\tag{79}
$$

become all pairwise equivalent through such transformations, because one is allowed to replace $\underline{z}_1, \ldots, \underline{z}_k$ by $-\underline{z}_1, \ldots, -\underline{z}_k$ without touching $z_1, \ldots, z_k$; even the factor $2i$ can be erased:

$$
w = \underline{w} + z_1\underline{z}_1 + \cdots + z_k\underline{z}_k + z_{k+1}\underline{z}_{k+1} + \cdots + z_n\underline{z}_n. \tag{80}
$$

Therefore, when passing to systems of partial differential equations associated with CR manifolds, *Levi form signatures drop*.

Consequently, when one applies the main theorem of [3], according to which a Levi nondegenerate $M \subset \mathbb{C}^{n+1}$ *having given Levi form signature* $(k, n-k)$ is pseudospherical if and only if (notation same as in [3]) its Hachtroudi system is equivalent to

$$w'_{z'_{k_1} z'_{k_2}}\left(z'\right) = 0 \quad \left(1 \leqslant k_1, k_2 \leqslant n\right) \tag{81}$$

and moreover if and only if—after translating back to the graphing function Θ the explicit condition of Hachtroudi—the identical vanishing property holds,

$$\begin{aligned}
0 \equiv \frac{1}{\Delta^3} & \left[\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \left[\Delta^{\mu}_{[0_{1+\ell_1}]} \right.\right. \\
& \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}^\tau} \right\} \\
& - \frac{\delta_{k_1,\ell_1}}{n+2} \sum_{\ell_3=1}^{n} \Delta^{\mu}_{[0_{1+\ell_3}]} \\
& \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell_3} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell_3} \partial z_{k_2} \partial \bar{t}^\tau} \right\} \\
& - \frac{\delta_{k_1,\ell_2}}{n+2} \sum_{\ell_3=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \\
& \cdot \Delta^{\nu}_{[0_{1+\ell_3}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell_3} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell_3} \partial z_{k_2} \partial \bar{t}^\tau} \right\} \\
& - \frac{\delta_{k_2,\ell_1}}{n+2} \sum_{\ell_3=1}^{n} \Delta^{\mu}_{[0_{1+\ell_3}]} \\
& \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell_3} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell_3} \partial \bar{t}^\tau} \right\} \\
& - \frac{\delta_{k_2,\ell_2}}{n+2} \sum_{\ell_3=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \\
& \left. \cdot \Delta^{\nu}_{[0_{1+\ell_3}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell_3} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell_3} \partial \bar{t}^\tau} \right\} \right] \\
& + \frac{1}{(n+1)(n+2)} \cdot \left[\delta_{k_1,\ell_1} \delta_{k_2,\ell_2} + \delta_{k_2,\ell_1} \delta_{k_1,\ell_2} \right] \\
& \cdot \sum_{\ell_3=1}^{n} \sum_{\ell_4=1}^{n} \Delta^{\mu}_{[0_{1+\ell_3}]} \cdot \Delta^{\nu}_{[0_{1+\ell_4}]} \left\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell_3} \partial z_{\ell_4} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}^\mu \bar{t}^\nu]} \right. \\
& \left.\left. \cdot \frac{\partial^3 \Theta}{\partial z_{\ell_3} \partial z_{\ell_4} \partial \bar{t}^\tau} \right\} \right] \quad \left(1 \leqslant k_1, k_2 \leqslant n;\ \left(1 \leqslant \ell_1, \ell_2 \leqslant n\right)\right),
\end{aligned} \tag{82}$$

one can reason exactly as in the preceding section for $M^3 \subset \mathbb{C}^2$ —noticing that the denominator is the same as $1/\Delta^3$, noticing that the numerator is similarly polynomial in the partial derivatives of the graphing function Θ —but when one jumps from a Levi nondegenerate point $p \in M \cap \square^{n+1}_{\rho_0}$ to another Levi nondegenerate point $q \in M \cap \square^{n+1}_{\rho_0}$, from the property of local equivalence at q to

$$w''_{z''_{k_1} z''_{k_2}}\left(z''\right) = 0 \quad \left(1 \leqslant k_1, k_2 \leqslant n\right), \tag{83}$$

one can only conclude that the complexification of M near q is equivalent near q to

$$\begin{aligned} w'' = {} & \underline{w}'' + z''_1 \underline{z}''_1 + \cdots + z''_k \underline{z}''_k + z''_{k+1} \underline{z}''_{k+1} + \cdots \\ & + z''_n \underline{z}''_n \end{aligned} \tag{84}$$

so that the Levi form signature can in principle change—and it really does in Example 6.3 of [1]—through Levi degenerate points.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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