C. R. Acad. Sci. Paris, Ser. I 356 (2018) 214–221

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Analytic geometry/Differential geometry

Parametric CR-umbilical locus of ellipsoids in $\mathbb{C}^2$

Détermination paramétrique du lieu CR-ombilic d'ellipsoïdes dans $\mathbb{C}^2$

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ARTICLE INFO

Article history:
Received 12 July 2017
Accepted after revision 13 November 2017
Available online 12 January 2018

Presented by the Editorial Board

ABSTRACT

For every real numbers $a \geqslant 1$, $b \geqslant 1$ with $(a, b) \neq (1, 1)$, the curve parametrized by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2 \cong \mathbb{R}^4$

$$\gamma: \quad \theta \longmapsto \big(x(\theta) + \sqrt{-1}\, y(\theta),\; u(\theta) + \sqrt{-1}\, v(\theta)\big)$$

with components:

$$x(\theta) := \sqrt{\tfrac{a-1}{a\,(ab-1)}}\,\cos\theta, \qquad y(\theta) := \sqrt{\tfrac{b\,(a-1)}{ab-1}}\,\sin\theta,$$
$$u(\theta) := \sqrt{\tfrac{b-1}{b\,(ab-1)}}\,\sin\theta, \qquad v(\theta) := -\sqrt{\tfrac{a\,(b-1)}{ab-1}}\,\cos\theta,$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \mathsf{UmbCR}(\mathsf{E}_{a,b}) \subset \mathsf{E}_{a,b}$$

of the ellipsoid $\mathsf{E}_{a,b} \subset \mathbb{C}^2$ of equation $a\,x^2 + y^2 + b\,u^2 + v^2 = 1$, where the CR-umbilical locus of a Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ is the set of points at which the Cartan curvature of M vanishes.

RÉSUMÉ

Pour tous nombres réels $a \geqslant 1$, $b \geqslant 1$ avec $(a, b) \neq (1, 1)$, la courbe paramétrée par $\theta \in \mathbb{R}$ à valeurs dans $\mathbb{C}^2 \cong \mathbb{R}^4$

$$\gamma: \quad \theta \longmapsto \big(x(\theta) + \sqrt{-1}\, y(\theta),\; u(\theta) + \sqrt{-1}\, v(\theta)\big)$$

ayant pour composantes :

$$x(\theta) := \sqrt{\tfrac{a-1}{a\,(ab-1)}}\,\cos\theta, \qquad y(\theta) := \sqrt{\tfrac{b\,(a-1)}{ab-1}}\,\sin\theta,$$
$$u(\theta) := \sqrt{\tfrac{b-1}{b\,(ab-1)}}\,\sin\theta, \qquad v(\theta) := -\sqrt{\tfrac{a\,(b-1)}{ab-1}}\,\cos\theta,$$

est d'image contenue dans le lieu CR-ombilic :

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https://doi.org/10.1016/j.crma.2017.11.019
1631-073X/

$$\gamma(\mathbb{R}) \subset \mathsf{UmbCR}\big(\mathsf{E}_{a,b}\big) \subset \mathsf{E}_{a,b}$$

de l'ellipsoïde $\mathsf{E}_{a,b} \subset \mathbb{C}^2$ d'équation $a\,x^2 + y^2 + b\,u^2 + v^2 = 1$, où le lieu CR-ombilic d'une hypersurface Levi non dégénérée $M^3 \subset \mathbb{C}^2$ est l'ensemble des points en lesquels la courbure de Cartan de M s'annule.

1. Introduction

In 1932, Élie Cartan [1–3] showed that a local real-analytic ($\mathscr{C}^\omega$) hypersurface $M^3 \subset \mathbb{C}^2$ is determined up to local biholomorphic equivalence by a single invariant function:

$$\mathfrak{I}^M_{\mathsf{Cartan}}\colon \quad M \longrightarrow \mathbb{C},$$

together with its (covariant) derivatives with respect to a certain coframe of differential 1-forms on an 8-dimensional principal bundle $P^8 \longrightarrow M$. In coordinates $(z, w) = \big(x + \sqrt{-1}\,y,\, u + \sqrt{-1}\,v\big)$ on $\mathbb{C}^2$, whenever M is:

- either a *complex graph:*

$$\big\{(z, w) \in \mathbb{C}^2\colon\ w = \Theta\big(z, \overline{z}, \overline{w}\big)\big\},$$

- or a *real graph:*

$$\big\{(z, w) \in \mathbb{C}^2\colon\ v = \varphi(x, y, u)\big\},$$

- or represented in *implicit form:*

$$\big\{(z, w) \in \mathbb{C}^2\colon\ \rho\big(z, w, \overline{z}, \overline{w}\big) = 0\big\},$$

it is known that $\mathfrak{I}^M_{\mathsf{Cartan}}$ depends on the respective 6-jets:

$$J^6_{z,\overline{z},\overline{w}}\,\Theta, \qquad J^6_{x,y,u}\,\varphi, \qquad J^6_{z,w,\overline{z},\overline{w}}\,\rho.$$

The *invariancy* of $\mathfrak{I}^M_{\mathsf{Cartan}}$ means that, for any local biholomorphism $h\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, setting $M' := h(M)$, it holds at every point $p \in M$ that:

$$\mathfrak{I}^{M'}_{\mathsf{Cartan}}\big(h(p)\big) = \nu(p)\,\mathfrak{I}^M_{\mathsf{Cartan}}(p), \qquad (\forall\, p \in M)$$

for some nowhere vanishing (local) function $\nu\colon M \longrightarrow \mathbb{C}\backslash\{0\}$. This guarantees that the locus of *CR-umbilical* points:

$$\mathsf{Umb}_{\mathsf{CR}}(M) := \big\{p \in M\colon\ \mathfrak{I}^M_{\mathsf{Cartan}}(p) = 0\big\}$$

is intrinsic. Furthermore, when M is connected, it is well known that $\mathsf{Umb}_{\mathsf{CR}}(M)$ contains an open set $\emptyset \neq V \subset M$ if and only if M is *spherical*, in the sense of being locally biholomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$.

In 1974, Chern and Moser [4] raised the problem whether $\emptyset \neq \mathsf{Umb}_{\mathsf{CR}}(M)$ for compact Levi nondegenerate $\mathscr{C}^\omega$ hypersurfaces $M^{2\mathrm{N}-1} \subset \mathbb{C}^{\mathrm{N}}$ when $\mathrm{N} \geqslant 2$. This paper studies the more specific aspects below.

Question 1. *Can $\mathsf{Umb}_{\mathsf{CR}}(M)$ be described explicitly?*

But because $\mathfrak{I}^M_{\mathsf{Cartan}}$ is '*complicated*', as confirmed in [8,9], the question is nontrivial even in simplest nonspherical examples, like, *e.g.*, the real ellipsoids introduced and studied by Webster in [11–13].

In $\mathbb{C}^{\mathrm{N}\geqslant 2} \cong \mathbb{R}^{2\mathrm{N}\geqslant 4}$ equipped with coordinates $z_i = x_i + \sqrt{-1}\,y_i$, an *ellipsoid* is the image of the unit sphere:

$$S^{2\mathrm{N}-1} := \big\{z \in \mathbb{C}^{\mathrm{N}}\colon\ |z_1|^2 + \cdots + |z_{\mathrm{N}}|^2 = 1\big\},$$

through a real affine transformation of $\mathbb{R}^{2\mathrm{N}}$; hence its equation takes the form:

$$\sum_{1\leqslant i\leqslant \mathrm{N}} \big(\alpha_i\,x_i^2 + \beta_i\,y_i^2\big) = 1, \qquad (\mathsf{E}_{\alpha,\beta})$$

with real constants $\alpha_i \geqslant \beta_i > 0$ — replace $z_i \longmapsto \sqrt{-1}\,z_i$ if necessary.

The complex geometry of ellipsoids (Segre varieties, dynamics) began in Webster's seminal article [11], in which it was verified that two ellipsoids $\mathsf{E}_{\alpha,\beta} \cong \mathsf{E}_{\alpha',\beta'}$ are biholomorphically equivalent if and only if up to permutation:

$$\frac{\alpha_i - \beta_i}{\alpha_i + \beta_i} = \frac{\alpha_i' - \beta_i'}{\alpha_i' + \beta_i'}. \qquad (1 \leqslant i \leqslant \mathrm{N})$$

Replacing $z_i \longmapsto \frac{1}{\sqrt{\beta_i}}\, z_i$ and setting $a_i := \frac{\alpha_i}{\beta_i}$, whence $1 \leqslant a_i$, leads to a convenient representation:

$$\sum_{1 \leqslant i \leqslant \mathrm{N}} \big(a_i\, x_i^2 + y_i^2\big) = 1. \qquad (\mathsf{E}_{a_1,\dots,a_{\mathrm{N}}})$$

Yet an alternative view, due to Webster in [12], is:

$$\sum_{1 \leqslant i \leqslant \mathrm{N}} \big(z_i \bar{z}_i + A_i\, (z_i^2 + \bar{z}_i^2)\big) = 1, \qquad (\mathsf{E}_{A_1,\dots,A_{\mathrm{N}}})$$

obtained by setting $A_i := \frac{a_i - 1}{2a_i + 2}$, whence $0 \leqslant A_i < \frac{1}{2}$, so that $a_i = \frac{1+2A_i}{1-2A_i}$, then by changing coordinates $z_i =: \sqrt{1-2A_i}\, z_i'$, and then by dropping primes.

In $\mathbb{C}^{\mathrm{N}}$ when $\mathrm{N} \geqslant 3$, what corresponds to the invariant $\mathfrak{I}_{\mathsf{Cartan}}^M$ is the Hachtroudi–Chern tensor $S_{\rho\sigma}^{\alpha\beta}$ with indices $1 \leqslant \alpha, \beta, \rho, \sigma \leqslant \mathrm{N}$, and the concerned CR-umbilical locus:

$$\mathsf{Umb}_{\mathsf{CR}}(M) := \big\{p \in M \colon\ S_{\alpha\rho}^{\beta\sigma}(p) = 0,\ \forall\, \alpha, \rho, \beta, \sigma\big\},$$

is known, through local biholomorphisms $h \colon \mathbb{C}^{\mathrm{N}} \longrightarrow \mathbb{C}^{\mathrm{N}}$ as above, to enjoy

$$h\big(\mathsf{Umb}_{\mathsf{CR}}(M)\big) = \mathsf{Umb}_{\mathsf{CR}}\big(h(M)\big).$$

Theorem 2. ([12]) *In $\mathbb{C}^{\mathrm{N} \geqslant 3}$, if $0 < A_1 < \cdots < A_{\mathrm{N}} < \frac{1}{2}$, then:*

$$\emptyset = \mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{A_1,\dots,A_{\mathrm{N}}}\big).$$

This motivated Huang–Ji in [5] to study the question for compact $\mathscr{C}^\omega$ hypersurfaces $M \subset \mathbb{C}^2$. If $M = \{\rho = 0\}$, the expected dimension of:

$$\mathsf{Umb}_{\mathsf{CR}}(M) = \big\{0 = \rho = \mathbb{R}e\, \mathfrak{I}_{\mathsf{Cartan}} = \Im\, \mathfrak{I}_{\mathsf{Cartan}}\big\}$$

should be $4 - 3 = 1$, although this is not rigorous, for $\mathbb{R}$ is not algebraically closed.

Theorem 3. (Implicitly proved in [5].) *Every real ellipsoid $\mathsf{E}_{a,b} \subset \mathbb{C}^2$ of equation:*

$$a\, x^2 + y^2 + b\, u^2 + v^2 = 1 \qquad (a \geqslant 1,\ b \geqslant 1,\ (a,b) \neq (1,1))$$

enjoys:

$$\dim_{\mathbb{R}} \mathsf{Umb}_{\mathsf{CR}}(M) \geqslant 1.$$

In other words, it contains at least some (real algebraic) curve. This article finds an explicit curve.

Theorem 4. *For every real numbers $a \geqslant 1$, $b \geqslant 1$ with $(a, b) \neq (1, 1)$, the curve parametrized by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\gamma \colon \quad \theta \longmapsto \big(x(\theta) + \sqrt{-1}\, y(\theta),\ u(\theta) + \sqrt{-1}\, v(\theta)\big)$$

with components:

$$x(\theta) := \sqrt{\tfrac{a-1}{a\,(ab-1)}}\, \cos\theta, \qquad y(\theta) := \sqrt{\tfrac{b\,(a-1)}{ab-1}}\, \sin\theta,$$

$$u(\theta) := \sqrt{\tfrac{b-1}{b\,(ab-1)}}\, \sin\theta, \qquad v(\theta) := -\sqrt{\tfrac{a\,(b-1)}{ab-1}}\, \cos\theta,$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{a,b}\big) \subset \mathsf{E}_{a,b}$$

of the ellipsoid $\mathsf{E}_{a,b} \subset \mathbb{C}^2$ of equation $a\, x^2 + y^2 + b\, u^2 + v^2 = 1$.

In other words:

$$\mathfrak{J}_{\mathsf{Cartan}}^{\mathsf{E}_{a,b}}\big(\gamma(\theta)\big) = 0. \qquad (\forall\theta\in\mathbb{R})$$

As is known for ellipsoids, Cartan's invariant $\mathfrak{J}_{\mathsf{Cartan}}^{\mathsf{E}_{a,b}}$ exhibits a high complexity, *e.g.*, $\sim 40\,000$ terms in [8]. So this theorem might be interpreted as a somewhat unexpectedly nice and simple description of $\mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{a,b}\big)$.

All computations of this paper were done by hand on a computer, which shows dim $\mathsf{UmbCR}\big(\mathsf{E}_{a,b}\big) = 1$.

2. Explicit expression of Cartan's CR-Invariant $\mathfrak{J}$

In $\mathbb{C}^2$ equipped with coordinates $(z,w) = \big(x+\sqrt{-1}\,y,\ u+\sqrt{-1}\,v\big)$, consider a connected real-analytic ($\mathscr{C}^\omega$) 3-dimensional hypersurface:

$$M^3 := \big\{(z,w)\in\mathbb{C}^2\colon\ \rho(z,w,\overline{z},\overline{w}) = 0\big\},$$

with $\overline{\rho} = \rho$, and with $\mathrm{d}\rho\big|_M$ never zero. Local or global M, compact or open, bounded or unbounded, can be equally treated.

The two vector fields:

$$L := -\,\rho_w\,\frac{\partial}{\partial z} + \rho_z\,\frac{\partial}{\partial w} \qquad \text{and} \qquad \overline{L} := -\,\rho_{\overline{w}}\,\frac{\partial}{\partial \overline{z}} + \rho_{\overline{z}}\,\frac{\partial}{\partial \overline{w}}$$

generate $T^{1,0}M$ and $T^{0,1}M$.

If $h\colon\mathbb{C}^2\longrightarrow\mathbb{C}^2$ is a local biholomorphism:

$$(z,w) \longmapsto \big(f(z,w), g(z,w)\big) =: (z', w'),$$

if $M = \{\rho = 0\}$ and $M' = \{\rho' = 0\}$ are two $\mathscr{C}^\omega$ hypersurfaces; if $h(M)\subset M'$, there is a nowhere vanishing function $\mu\colon M\longrightarrow \mathbb{C}\backslash\{0\}$ such that:

$$\mu(z,w,\overline{z},\overline{w})\,\rho(z,w,\overline{z},\overline{w}) \;\equiv\; \rho'\big(f(z,w), g(z,w), \overline{f}(\overline{z},\overline{w}), \overline{g}(\overline{z},\overline{w})\big),$$

whence in $\mathbb{C}\{z,w,\overline{z},\overline{w}\}$:

$$\mu\Big(-\,\rho_w\,\frac{\partial}{\partial z} + \rho_z\,\frac{\partial}{\partial w}\Big) \;=\; \big(f_z g_w - f_w g_z\big)\Big(-\,\rho'_{w'}\,\frac{\partial}{\partial z'} + \rho'_{z'}\,\frac{\partial}{\partial w'}\Big).$$

Furthermore, the *Levi determinant:*

$$\begin{aligned}\mathsf{L}(\rho) &:= -\begin{vmatrix} 0 & \rho_z & \rho_w \\ \rho_{\overline{z}} & \rho_{z\overline{z}} & \rho_{w\overline{z}} \\ \rho_{\overline{w}} & \rho_{z\overline{w}} & \rho_{w\overline{w}} \end{vmatrix} \\ &= \rho_{\overline{z}}\rho_z\rho_{w\overline{w}} - \rho_{\overline{z}}\rho_w\rho_{z\overline{w}} - \rho_{\overline{w}}\rho_z\rho_{\overline{z}w} + \rho_{\overline{w}}\rho_w\rho_{z\overline{z}},\end{aligned}$$

enjoys:

$$\mu^3\,\mathsf{L}(\rho) = \big(f_z g_w - f_w g_z\big)\big(\overline{f}_{\overline{z}}\overline{g}_{\overline{w}} - \overline{f}_{\overline{w}}\overline{g}_{\overline{z}}\big)\,\mathsf{L}(\rho'). \qquad (\text{on}\,M)$$

Definition 5. A smooth hypersurface $M^3\subset\mathbb{C}^2$ is called *Levi nondegenerate* at a point $p\in M$ if:

$$0 \neq \mathsf{L}(p).$$

From now on, all M will be assumed smooth and Levi nondegenerate at every point, without further mention.

When $0\neq\rho_w(p) = \rho_{\overline{w}}(p)$ at a point $p = (z_p, w_p)\in M$, the implicit function theorem represents M as a complex graph:

$$w = \Theta\big(z,\overline{z},\overline{w}\big) \qquad \text{or equivalently:} \qquad \overline{w} = \overline{\Theta}\big(\overline{z}, z, w\big),$$

in terms of a $\mathscr{C}^\omega$ defining function Θ. A similar graphed representation exists at points $q = (z_q, w_q)\in M$ at which $0\neq\rho_z(q) = \rho_{\overline{z}}(q)$.

Differentiating the identity:

$$0 \equiv \rho\big(z, \Theta\big(z,\overline{z},\overline{w}\big), \overline{z}, \overline{w}\big), \qquad (\text{in}\,\mathbb{C}\{z,\overline{z},\overline{w}\})$$

once with respect to z, $\overline{z}$, $\overline{w}$ yields:

$$0 \equiv \rho_z + \Theta_z \, \rho_w,$$
$$0 \equiv \rho_{\overline{z}} + \Theta_{\overline{z}} \, \rho_w,$$
$$0 \equiv \rho_{\overline{w}} + \Theta_{\overline{w}} \, \rho_w,$$

and next twice with respect to zz, $z\overline{z}$, $z\overline{w}$, $\overline{z}\overline{z}$, $\overline{z}\overline{w}$, $\overline{w}\overline{w}$ gives:

$$\begin{aligned}
0 &\equiv \rho_{zz} + 2\,\Theta_z \, \rho_{zw} + \Theta_z \, \Theta_z \, \rho_{ww} + \Theta_{zz} \, \rho_w, \\
0 &\equiv \rho_{z\overline{z}} + \Theta_z \, \rho_{\overline{z}w} + \Theta_{\overline{z}} \, \rho_{zw} + \Theta_z \, \Theta_{\overline{z}} \, \rho_{ww} + \Theta_{z\overline{z}} \, \rho_w, \\
0 &\equiv \rho_{z\overline{w}} + \Theta_z \, \rho_{w\overline{w}} + \Theta_{\overline{w}} \, \rho_{zw} + \Theta_z \, \Theta_{\overline{w}} \, \rho_{ww} + \Theta_{z\overline{w}} \, \rho_w, \\
0 &\equiv \rho_{\overline{z}\overline{z}} + 2\,\Theta_{\overline{z}} \, \rho_{\overline{z}w} + \Theta_{\overline{z}} \, \Theta_{\overline{z}} \, \rho_{ww} + \Theta_{\overline{z}\overline{z}} \, \rho_w, \\
0 &\equiv \rho_{\overline{z}\overline{w}} + \Theta_{\overline{z}} \, \rho_{w\overline{w}} + \Theta_{\overline{w}} \, \rho_{\overline{z}w} + \Theta_{\overline{z}} \, \Theta_{\overline{w}} \, \rho_{ww} + \Theta_{\overline{z}\overline{w}} \, \rho_w, \\
0 &\equiv \rho_{\overline{w}\overline{w}} + 2\,\Theta_{\overline{w}} \, \rho_{w\overline{w}} + \Theta_{\overline{w}} \, \Theta_{\overline{w}} \, \rho_{ww} + \Theta_{\overline{w}\overline{w}} \, \rho_w.
\end{aligned} \tag{6}$$

It holds that:

$$\{\rho_w \neq 0\} = \{\Theta_{\overline{w}} \neq 0\}. \qquad (\text{in } M)$$

Definition 7. Call M *spherical* if it is locally biholomorphic to:

$$S^3 := \{(z, w) \in \mathbb{C}^2 : \; z\overline{z} + w\overline{w} = 1\}.$$

When M is connected, the principle of analytic continuation guarantees the propagation of this property. Next, set:

$$\Delta := -\Theta_{\overline{w}} \, \Theta_{z\overline{z}} + \Theta_{\overline{z}} \, \Theta_{z\overline{w}}.$$

Lemma 8. *At a point* $p \in \{\Theta_{\overline{w}} \neq 0\}$:

$$M \text{ is Levi nondegenerate at } p \iff \Delta(p) \neq 0.$$

Levi's nondegeneracy being a biholomorphically invariant feature, spherical M are so since S^3 is.

A characterization of sphericity in terms of some defining function for a hypersurface $M^3 \subset \mathbb{C}^2$ is as follows (*cf.* also [6,10,5]). To recall it, set:

$$\square := \frac{\Delta}{-\Theta_{\overline{w}}},$$

and use instead:

$$\begin{aligned}
\overline{\mathscr{L}} &:= -\frac{1}{\rho_{\overline{w}}} \, \overline{L} \\
&= \frac{\partial}{\partial \overline{z}} - \frac{\Theta_{\overline{z}}}{\Theta_{\overline{w}}} \, \frac{\partial}{\partial \overline{w}}.
\end{aligned}$$

Theorem 9. ([7]) *At a point* $p \in \{\Theta_{\overline{w}} \neq 0\}$*, the hypersurface* M *is spherical if and only if, near* p*:*

$$0 \equiv \frac{1}{\square} \, \overline{\mathscr{L}}\bigg(\frac{1}{\square} \, \overline{\mathscr{L}}\bigg(\frac{1}{\square} \, \overline{\mathscr{L}}\bigg(\frac{1}{\square} \, \overline{\mathscr{L}}\big(\Theta_{zz}\big)\bigg)\bigg)\bigg).$$

Exchanging $z \longleftrightarrow w$ yields a similar formula at points $q \in \{\rho_z \neq 0\}$.

Corollary 10. *In* $\{\rho_w \neq 0\} = \{\Theta_{\overline{w}} \neq 0\}$*, a partly expanded characterization of sphericity is:*

$$\begin{aligned} 0 \equiv\ & \frac{\overline{\mathscr{L}}^4(\Theta_{zz})}{\square^4} - \\ & - 6\,\frac{\overline{\mathscr{L}}(\square)\,\overline{\mathscr{L}}^3(\Theta_{zz})}{\square^5} - 4\,\frac{\overline{\mathscr{L}}^2(\square)\,\overline{\mathscr{L}}^2(\Theta_{zz})}{\square^5} - \frac{\overline{\mathscr{L}}^3(\square)\,\overline{\mathscr{L}}(\Theta_{zz})}{\square^5} + \\ & + 15\,\frac{\big[\overline{\mathscr{L}}(\square)\big]^2\,\overline{\mathscr{L}}^2(\Theta_{zz})}{\square^6} + 10\,\frac{\overline{\mathscr{L}}(\square)\,\overline{\mathscr{L}}^2(\square)\,\overline{\mathscr{L}}(\Theta_{zz})}{\square^6} - \\ & - 15\,\frac{\big[\overline{\mathscr{L}}(\square)\big]^3\,\overline{\mathscr{L}}(\Theta_{zz})}{\square^7}. \end{aligned}$$

Without presenting details, it is known that Cartan's treatment of the concerned biholomorphic equivalence problem brings a single invariant function:

$$\mathfrak{I}^M_{\mathsf{Cartan}} \colon \quad M \longrightarrow \mathbb{C},$$

other invariants being (covariant) derivations of it, and that:

$$M \text{ is spherical} \quad \Longleftrightarrow \quad 0 \equiv \mathfrak{I}^M_{\mathsf{Cartan}}.$$

Notation 11. For two functions $I_1 \colon M \longrightarrow \mathbb{C}$ and $I_2 \colon M \longrightarrow \mathbb{C}$, write:

$$I_2 \doteqdot I_1,$$

when there is a nowhere vanishing function $\mu \colon M \longrightarrow \mathbb{C}\backslash\{0\}$ such that:

$$I_2 = \mu\, I_1.$$

For instance:

$$\mathfrak{I}^M_{\mathsf{Cartan}} \doteqdot \left(\frac{1}{\square}\,\overline{\mathscr{L}}\right)^4 \big(\Theta_{zz}\big).$$

Now, translate the formula of Corollary 10 to the case where M is given in implicit representation:

$$0 = \rho\big(z, w, \overline{z}, \overline{w}\big).$$

Set:

$$\mathsf{H}(\rho) := \rho_z\rho_z\,\rho_{ww} - 2\,\rho_z\rho_w\,\rho_{zw} + \rho_w\rho_w\,\rho_{zz},$$

with on $\{\rho_w \neq 0\}$:

$$\Theta_{zz} = -\,\frac{\mathsf{H}(\rho)}{\rho_w\rho_w\rho_w}.$$

Remind the *Levi determinant:*

$$\mathsf{L}(\rho) := \rho_{\overline{z}}\rho_z\rho_{w\overline{w}} - \rho_{\overline{z}}\rho_w\rho_{z\overline{w}} - \rho_{\overline{w}}\rho_z\rho_{\overline{z}w} + \rho_{\overline{w}}\rho_w\rho_{z\overline{z}},$$

which satisfies on $\{\rho_w \neq 0\}$:

$$\mathsf{L}(\rho) \doteqdot \Delta,$$

i.e. more precisely thanks to (6):

$$\mathsf{L}(\rho) = -\,\rho_w\,\rho_w\,\rho_w\,\Delta.$$

Corollary 12. *On* $\{\rho_w \neq 0\}$*, up to a nowhere vanishing function:*

$$\mathfrak{I}^M_{\mathsf{Cartan}} \doteqdot I_{[w]},$$

where:

$$
\begin{aligned}
I_{[w]} \;:=\; 12\,(\rho_w)^9 \bigg\{ & \bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^3 \overline{L}^4\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \\
& - 6\bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^3\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - 4\bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}^2\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^2\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}^3\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) + \\
& + 15\,\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg[\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\bigg]^2 \overline{L}^2\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) + 10\,\frac{\mathsf{L}(\rho)}{\rho_w^2}\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^2\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \\
& - 15\bigg[\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\bigg]^3 \overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg)\bigg\}.
\end{aligned}
$$

Furthermore, exchanging $z \longleftrightarrow w$, there is an *exact* formal coincidence:

$$I_{[z]} = I_{[w]},$$

although this is not needed here.

3. Pullback to an exceptional curve on an ellipsoid

To prove Theorem 4, it suffices to verify that:

$$0 \overset{?}{=} \gamma^*\big(I_{[w]}\big)(\theta). \qquad (\forall \theta \in \mathbb{R})$$

Drop the factor $12\,(\rho_w)^9 \doteqdot 1$, and call T_1, T_2, T_3, T_4, T_5, T_6, T_7 the seven concerned terms, so that the goal becomes:

$$0 \overset{?}{=} \gamma^*(T_1) + \gamma^*(T_2) + \gamma^*(T_3) + \gamma^*(T_4) + \gamma^*(T_5) + \gamma^*(T_6) + \gamma^*(T_7).$$

Hand computations provide formulas of the shape:

$$
\begin{aligned}
T_1 &= \tfrac{1}{8}\sqrt{-1}\,(a-1)\,\frac{N_1}{D}, \\
T_2 &= \tfrac{3}{4}\sqrt{-1}\,(a-1)\,\frac{N_2}{D}, \\
T_3 &= \tfrac{1}{2}\sqrt{-1}\,(a-1)\,\frac{N_3}{D}, \\
T_4 &= \tfrac{1}{8}\sqrt{-1}\,(a-1)\,\frac{N_4}{D}, \\
T_5 &= \tfrac{15}{8}\sqrt{-1}\,(a-1)\,\frac{N_5}{D}, \\
T_6 &= \tfrac{5}{4}\sqrt{-1}\,(a-1)\,\frac{N_6}{D}, \\
T_7 &= \tfrac{15}{8}\sqrt{-1}\,(a-1)\,\frac{N_7}{D},
\end{aligned}
$$

with, in denominator place:

$$D := \Big(\sqrt{a}\,\cos\theta - \sqrt{-1}\,\sqrt{b}\,\sin\theta\Big)^8\,(a\,b-1)\,\bigg(\frac{b-1}{a\,b-1}\bigg)^{\frac{11}{2}},$$

with numerator 1:

$$
\begin{aligned}
N_1 := {}& \cos^7\theta\,\Big[499\,a^{9/2}b^3 + 625\,a^{9/2}b^2 - 233\,a^{7/2}b^3 + 205\,a^{9/2}b - 631\,a^{7/2}b^2 + 15\,a^{9/2} - 415\,a^{7/2}b - 65\,a^{7/2}\Big] \\
& + \sqrt{-1}\,\cos^6\theta\,\sin\theta\,\Big[2887\,a^4b^{7/2} + 4401\,a^4b^{5/2} - 1297\,a^3b^{7/2} + 1905\,a^4b^{3/2} - 4059\,a^3b^{5/2} + 215\,a^4b^{1/2} - 3327\,a^3b^{3/2} - 725\,a^3b^{1/2}\Big] \\
& + \cos^5\theta\,\sin^2\theta\,\Big[-7023\,a^{7/2}b^4 - 13021\,a^{7/2}b^3 + 3013\,a^{5/2}b^4 - 7105\,a^{7/2}b^2 + 11011\,a^{5/2}b^3 - 1075\,a^{7/2}b + 11059\,a^{5/2}b^2 + 3141\,a^{5/2}b\Big] \\
& + \sqrt{-1}\,\cos^4\theta\,\sin^3\theta\,\Big[-9267\,a^3b^{9/2} - 20989\,a^3b^{7/2} + 3757\,a^2b^{9/2} - 14101\,a^3\,b^{5/2} + 16279\,a^2b^{7/2} - 2683\,a^3b^{3/2} + 19891\,a^2b^{5/2} + 7113\,a^2b^{3/2}\Big] \\
& + \cos^3\theta\,\sin^4\theta\,\Big[7113\,a^{5/2}b^5 + 19891\,a^{5/2}b^4 - 2683\,a^{3/2}b^5 + 16279\,a^{5/2}b^3 - 14101\,a^{3/2}b^4 + 3757\,a^{5/2}b^2 - 20989\,a^{3/2}b^3 - 9267\,a^{3/2}b^2\Big] \\
& + \sqrt{-1}\,\cos^2\theta\,\sin^5\theta\,\Big[3141\,a^2b^{11/2} + 11059\,a^2b^{9/2} - 1075\,ab^{11/2} + 11011\,a^2b^{7/2} - 7105\,ab^{9/2} + 3013\,a^2b^{5/2} - 13021\,ab^{7/2} - 7023\,ab^{5/2}\Big] \\
& + \cos^1\theta\,\sin^6\theta\,\Big[-725\,a^{3/2}b^6 - 3327\,a^{3/2}b^5 + 215\,a^{1/2}b^6 - 4059\,a^{3/2}b^4 + 1905\,a^{1/2}b^5 - 1297\,a^{3/2}b^3 + 4401\,a^{1/2}b^4 + 2287\,a^{1/2}b^3\Big] \\
& + \sqrt{-1}\,\sin^7\theta\,\Big[-65\,ab^{13/2} - 415\,ab^{11/2} + 15\,b^{13/2} - 631\,ab^{9/2} + 205\,b^{11/2} - 233\,ab^{7/2} + 625\,b^{9/2} + 499\,b^{7/2}\Big],
\end{aligned}
$$

and with similar N_2, N_3, N_4, N_5, N_6, N_7, the expressions of which can be found in the announcement arxiv.org/abs/ of this publication.

End of proof of Theorem 4. The sum:

$$\frac{1}{8}N_1(\theta)+\frac{3}{4}N_2(\theta)+\frac{1}{2}N_3(\theta)+\frac{1}{8}N_4(\theta)+\frac{15}{8}N_5(\theta)+\frac{5}{4}N_6(\theta)+\frac{15}{8}N_7(\theta)=0,$$

is checked on a computer to be identically null.

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