Rigid Biholomorphic Equivalences [1] of Rigid $\mathfrak{C}_{2,1}$ Hypersurfaces $M^5 \subset \mathbb{C}^3$

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ABSTRACT. We study the local equivalence problem for real-analytic ($\mathscr{C}^\omega$) hypersurfaces $M^5 \subset \mathbb{C}^3$ which, in some holomorphic coordinates $(z_1, z_2, w) \in \mathbb{C}^3$ with $w = u + \sqrt{-1}v$, are *rigid* in the sense that their graphing functions:

$$u = F(z_1, z_2, \overline{z}_1, \overline{z}_2)$$

are independent of v. Specifically, we study the group $\mathsf{Hol}_{\mathsf{rigid}}(M)$ of *rigid* local biholomorphic transformations of the form:

$$(z_1, z_2, w) \longmapsto \big(f_1(z_1, z_2),\ f_2(z_1, z_2),\ aw + g(z_1, z_2)\big),$$

where $a \in \mathbb{R}\backslash\{0\}$ and $\frac{D(f_1,f_2)}{D(z_1,z_2)} \neq 0$, which preserve rigidity of hypersurfaces.

After performing a Cartan-type reduction to an appropriate $\{e\}$-structure, we find exactly *two* primary invariants I_0 and V_0, which we express explicitly in terms of the 5-jet of the graphing function F of M. The identical vanishing $0 \equiv I_0(J^5F) \equiv V_0(J^5F)$ then provides a necessary and sufficient condition for M to be locally *rigidly-biholomorphic* to the known model hypersurface:

$$M_{\mathsf{LC}}: \qquad u = \frac{z_1\overline{z}_1 + \frac{1}{2}z_1^2\overline{z}_2 + \frac{1}{2}\overline{z}_1^2 z_2}{1 - z_2\overline{z}_2}.$$

We establish that $\dim \mathsf{Hol}_{\mathsf{rigid}}(M) \leqslant 7 = \dim \mathsf{Hol}_{\mathsf{rigid}}(M_{\mathsf{LC}})$ always.

If one of these two primary invariants $I_0 \not\equiv 0$ or $V_0 \not\equiv 0$ does not vanish identically, then on either of the two Zariski-open sets $\{p \in M : I_0(p) \neq 0\}$ or $\{p \in M : V_0(p) \neq 0\}$, we show that this rigid equivalence problem between rigid hypersurfaces reduces to an equivalence problem for a certain 5-dimensional $\{e\}$-structure on M, that is, we get an invariant absolute parallelism on M^5. Hence $\dim \mathsf{Hol}_{\mathsf{rigid}}(M)$ drops from 7 to 5, illustrating the *gap phenomenon.*

Dedicated to Alexander Isaev†, in memoriam

1. Introduction

In 1907, Poincaré [31] gave a heuristic counting argument to show that real analytic hypersurfaces in $\mathbb{C}^2$ possess infinitely many *local invariants* under biholomorphic transformations. This gives rise to the classification problem of real submanifolds in complex spaces, which still occupies a central place in CR geometry.

The equivalence problem for *Levi nondegenerate* hypersurfaces in $\mathbb{C}^2$ was first solved by Élie Cartan [1, 2], as an application of his powerful *method of equivalence*, rooted in the « *Méthode du repère mobile* » of Darboux-Ribaucour.

In a landmark paper of 1974, Chern-Moser [5] successfully solved the equivalence problem for *Levi nondegenerate* hypersurfaces in $\mathbb{C}^n$ in any dimension $n \geqslant 2$, by applying Cartan's method of equivalence, as well as Poincaré's method of normal forms. For a treatment of both methods, we refer to the book of Jacobowitz [19] and the references therein. Other approaches to solving the equivalence problem in CR Geometry were also developed earlier by Tanaka [37], and later by Čap-Slovák [3] using parabolic geometry.

The classification and equivalence problems of *Levi degenerate* hypersurfaces in complex spaces are much less understood than the non-degenerate case. The task of identifying suitable *higher non-degeneracy* conditions was first considered by Freeman [11], and the modern language speaks of '2-*nondegeneracy*'. An elementary self-contained presentation

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of foundational aspects is available in [28], and will be enough for our purposes in this paper.

The appropriate setting for equivalence problem of degenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ has been determined to be the class of real analytic 5-dimensional 2-nondegenerate real hypersurfaces in $\mathbb{C}^3$ of constant Levi rank 1, which we denote by $\mathfrak{C}_{2,1}$ using the notation of Fels-Kaup. It is intermediate between the well-understood class of products of 3-dimensional hypersurfaces and $\mathbb{C}$, and the class of general 5-dimensional hypersurfaces in $\mathbb{C}^3$. First investigations of this class started in the late 1990's, and in 2008, Fels-Kaup [7] gave a complete classification of homogenous models. Merker-Nurowski [25, 26] recently extended these results to the more general para-CR context, *cf.* also [29].

For this $\mathfrak{C}_{2,1}$ class, an $\{e\}$-structure was constructed by Isaev-Zaitsev [17] and even better, a Cartan connection by Medori-Spiro [21, 22] — for not necessarily embedded CR manifolds. Effective computations have been conducted independently in the thesis [30, 27] of Pocchiola, who 'discovered' two explicit primary invariants $\boldsymbol{W}_0$ and $\boldsymbol{J}_0$, now known as *Pocchiola invariants*. Foo-Merker [8] completed the $\{e\}$-structure, and confirmed the existence of these invariants.

We also refer the readers to our recent treatment [9] of the classification problem for this class by Moser's method of normal form. As for degenerate hypersurfaces in $\mathbb{C}^n, n \geqslant 4$, the classification problem is widely open, and to the best of our knowledge, the works of Porter [32], and Porter-Zelenko [33] are the only treatments of the equivalence problem of 7-dimensional hypersurfaces in $\mathbb{C}^4$, and higher dimensional cases seem to be completely unexplored.

In this paper, we solve the equivalence problem for a special class of real analytic 5-dimensional 2-nondegenerate *rigid* hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1 under the action of *rigid* transformations, and we obtain two primary invariants. This is the first step towards a solution to the classification problem for this class of hypersurfaces, which is accomplished in our joint paper with Chen [4]. The class of *rigid* hypersurfaces was introduced by the late Isaev in his investigations [13, 14, 15, 16] of Pocchiola's invariants. This allows for a lot of simplifications in comparison with the case of general Levi degenerate hypersurfaces, while still giving rise to an interesting theory. There are also studies on rigid CR manifolds in other settings, such as rigid Levi non-degenerate real hypersurfaces in $\mathbb{C}^{n+1}$ by Stanton [35, 36] and rigid spheres by Ezhov-Schmalz [6].

More precisely, a hypersurface M^5 in $\mathbb{C}^3$ with coordinates $(z_1, z_2, w = u + i\,v)$ is called *rigid* if there is a vector field of the form $T = X + \overline{X}$ tangent to M, where X is a nonzero holomorphic vector field, such that $TM = T^c M \oplus \mathbb{R}T$. One can apply a local biholomorphic straightening transformation to obtain $X = i\,\frac{\partial}{\partial w}$ and $X + \overline{X} = 2\,\frac{\partial}{\partial v}$. It follows that M can be written as graph:

$$M^5\colon \quad u \,=\, F(z_1, z_2, \overline{z}_1, \overline{z}_2),$$

with a $\mathscr{C}^\omega$ function F independent of v. The rigid equivalence problem is studied under the action of *rigid biholomorphic transformations*, of the form:

$$(z_1, z_2, w) \;\longmapsto\; \big(f(z_1, z_2),\, g(z_1, z_2),\, a\,w + h(z_1, z_2)\big),$$

where f, g, h are holomorphic, independently of w, and where $a \in \mathbb{R}^*$. Section **2** gives more details.

Our main tool is *Cartan's method of equivalence*, and we refer our readers to [34, 18] for a presentation. For a given CR manifold M, this method constructs a principal bundle $\pi\colon P \longrightarrow M$ and a coframe of everywhere linearly independent 1-forms $\theta^1, \dots, \theta^{\dim P}$ on P such that:

(1) for any other CR manifold M', every CR diffeomorphism $\Phi\colon M \longrightarrow M'$ lifts uniquely to a diffeomorphism $\Pi\colon P \longrightarrow P'$ satisfying $\Pi^*{\theta'}^i = \theta^i$ for $1 \leqslant i \leqslant \mathsf{dim}\, P$, where P' and the ${\theta'}^i$'s are also constructed from M' by Cartan's method of equivalence;

(2) conversely, every diffeomorphism $\Pi\colon P \longrightarrow P'$ commuting with projections π, π' whose horizontal part is a diffeomorphims $\Phi : M \longrightarrow M'$:

$$\begin{tikzcd} P \arrow[r, "\Pi"] \arrow[d, "\pi"'] & P' \arrow[d, "\pi'"] \\ M \arrow[r, "\Phi"'] & M' \end{tikzcd}$$

which also satisfies $\Pi^*{\theta'}^i = \theta^i$ for $1 \leqslant i \leqslant \mathsf{dim}\, P$, has a horizontal part Φ a CR diffeomorphism.

In practice, as is the case in this paper, Cartan's method of equivalence is computationally *intensive*. Carrying out the method is a long, demanding and nontrivial task.

The very first step in investigating equivalence problems is to determine the homogeneous models for the class under consideration. In our case, it is the well-known *tube over the future light cone*:

$$\mathsf{Re}(z_1)^2 + \mathsf{Re}(z_2)^2 \,=\, u^2 \qquad\qquad (u>0),$$

whose Lie algebra was determined by Gaussier-Merker [10], and Fels-Kaup [7] to be the 10-dimensional $\mathfrak{so}(2,3,\mathbb{R})$. The following equivalent hypersurface, discovered by Gaussier-Merker [10], will be more useful for our purpose:

$$M_{\mathsf{LC}}\colon \qquad u = \frac{z_1\overline{z}_1 + \frac{1}{2}(z_1^2\overline{z}_2 + \overline{z}_1^2 z_2)}{1 - z_2\overline{z}_2}.$$

The Lie algebra of CR infinitesimal *rigid* automorphism will be determined in Section **2**. Our first result is the following:

Theorem 1.1. *The equivalence problem under local rigid biholomorphisms of $\mathscr{C}^\omega$ rigid real hypersurfaces $\{u = F(z_1, z_2, \overline{z}_1, \overline{z}_2)\}$ in $\mathbb{C}^3$ whose Levi form has constant rank 1 and which are everywhere 2-nondegenerate reduces to classifying $\{e\}$-structures on the 7-dimensional bundle $M^5 \times \mathbb{C}$ equipped with coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v, \mathsf{c}, \overline{\mathsf{c}})$ together with a coframe of 7 differential 1-forms:*

$$\{\rho,\ \kappa,\ \zeta,\ \overline{\kappa},\ \overline{\zeta},\ \alpha,\ \overline{\alpha}\},$$

which satisfy invariant structure equations of the shape:

$$\begin{aligned}
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + \sqrt{-1}\kappa \wedge \overline{\kappa},\\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \overline{\kappa},\\
d\zeta &= (\alpha - \overline{\alpha}) \wedge \zeta + \frac{1}{\mathsf{c}}\boldsymbol{I}_0\,\kappa \wedge \zeta + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{V}_0\,\kappa \wedge \overline{\kappa},\\
d\alpha &= \zeta \wedge \overline{\zeta} - \frac{1}{\mathsf{c}}\boldsymbol{I}_0\,\zeta \wedge \overline{\kappa} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{Q}_0\,\kappa \wedge \overline{\kappa} + \frac{1}{\overline{\mathsf{c}}}\overline{\boldsymbol{I}}_0\,\overline{\zeta} \wedge \kappa,
\end{aligned}$$

conjugate equations for $d\overline{\kappa}$, $d\overline{\zeta}$, $d\overline{\alpha}$ being understood.

The two primary invariants are explicitly given by

$$
\begin{aligned}
\boldsymbol{I}_0 &:= -\frac{1}{3}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k})))}{\overline{\mathscr{L}}_1(\boldsymbol{k})^2} + \frac{1}{3}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(\boldsymbol{k}))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})^3} \\
&\quad + \frac{2}{3}\frac{\mathscr{L}_1(\mathscr{L}_1(\overline{\boldsymbol{k}}))}{\mathscr{L}_1(\overline{\boldsymbol{k}})} + \frac{2}{3}\frac{\mathscr{L}_1(\overline{\mathscr{L}_1}(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}, \\
\boldsymbol{V}_0 &:= -\frac{1}{3}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k})))}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{5}{9}\left(\frac{\overline{\mathscr{L}_1}(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right)^2 - \\
&\quad - \frac{1}{9}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))\overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{1}{3}\overline{\mathscr{L}}_1(\overline{\boldsymbol{P}}) - \frac{1}{9}\overline{\boldsymbol{P}}\overline{\boldsymbol{P}},
\end{aligned}
$$

while the secondary invariant is

$$
\boldsymbol{Q}_0 := -\frac{1}{2}\overline{\mathscr{L}}_1(\boldsymbol{I}_0) + \frac{1}{3}\left(\boldsymbol{P} - \frac{\mathscr{L}_1(\mathscr{L}_1(\overline{\boldsymbol{k}}))}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\right)\bar{\boldsymbol{I}}_0 + \frac{1}{6}\left(\overline{\boldsymbol{P}} - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right)\boldsymbol{I}_0 + \frac{1}{2}\frac{\mathscr{K}(\mathbf{V}_0)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}.
$$

It will be shown that $\boldsymbol{Q}_0$ is real-valued, see equation (7.7). We refer the readers to the next section for the definitions of the vector fields $\{\mathscr{L}_1, \mathscr{K}\}$, and of the functions $\{\boldsymbol{P}, \boldsymbol{k}\}$.

Both $\boldsymbol{I}_0$ and $\boldsymbol{V}_0$ vanish identically for the Gaussier-Merker model M_{LC}, and it is a fundamental theorem in Cartan theory [34, 18] that the identical vanishing of all invariants provides constant coefficients Maurer-Cartan equations of a uniquely defined Lie group.

Theorem 1.2. *A 2-nondegenerate $\mathscr{C}^\omega$ constant Levi rank 1 local rigid hypersurface $M^5 \subset \mathbb{C}^3$ is rigidly biholomorphic to the model M_{LC} if and only if*

$$
0 \equiv \boldsymbol{I}_0 \equiv \boldsymbol{V}_0. \qquad \square
$$

A basis for the Maurer-Cartan forms on the local Lie group $\mathsf{Hol}_{\mathsf{rigid}}(M_{\mathsf{LC}})$ is provided by 7-differential 1-forms:

$$
\{\rho,\ \kappa,\ \zeta,\ \overline{\kappa},\ \overline{\zeta},\ \alpha,\ \overline{\alpha}\},
$$

where $\overline{\rho} = \rho$ is real, which enjoy the 7 structure equations with constant coefficients:

$$
\begin{aligned}
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + \sqrt{-1}\kappa \wedge \overline{\kappa}, && \\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \overline{\kappa}, & d\overline{\kappa} &= \overline{\alpha} \wedge \overline{\kappa} + \overline{\zeta} \wedge \kappa, \\
d\zeta &= (\alpha - \overline{\alpha}) \wedge \zeta, & d\overline{\zeta} &= (\overline{\alpha} - \alpha) \wedge \overline{\zeta}, \\
d\alpha &= \zeta \wedge \overline{\zeta}, & d\overline{\alpha} &= \overline{\zeta} \wedge \zeta.
\end{aligned}
$$

On the other hand, one can also obtain the same solution to the equivalence problem for M_{LC} using the vector fields method giving the Gaussier-Merker list.

Proposition 1.3. *For the model hypersurface:*

$$
M_{\mathsf{LC}}: \quad u = \frac{z_1\overline{z}_1 + \frac{1}{2}(z_1^2\overline{z}_2 + \overline{z}_1^2 z_2)}{1 - z_2\overline{z}_2},
$$

the Lie algebra $\mathfrak{hol}_{\mathsf{rigid}}(M_{\mathsf{LC}})$ *of infinitesimal rigid biholomorphisms is* 7*-dimensional, generated by:*

$$
\begin{aligned}
X^1 &= \sqrt{-1}\partial_w, \\
X^2 &= z_1\partial_{z_1} + 2w\partial_w, \\
X^3 &= \sqrt{-1}z_1\partial_{z_1} + 2\sqrt{-1}z_2\partial_{z_2}, \\
X^4 &= (z_2 - 1)\partial_{z_1} - 2z_1\partial_w, \\
X^5 &= (\sqrt{-1} + \sqrt{-1}z_2)\partial_{z_1} - 2\sqrt{-1}z_1\partial_w, \\
X^6 &= z_1z_2\partial_{z_1} + (z_2^2 - 1)\partial_{z_2} - z_1^2\partial_w, \\
X^7 &= \sqrt{-1}z_1z_2\partial_{z_1} + (\sqrt{-1}z_2^2 + \sqrt{-1})\partial_{z_2} - \sqrt{-1}z_1^2\partial_w.
\end{aligned}
$$

Let $(\partial_\rho, \partial_\kappa, \partial_\zeta, \partial_\alpha, \partial_{\overline{\kappa}}, \partial_{\overline{\zeta}}, \partial_{\overline{\alpha}})$ *be vector fields that are respective duals to the Maurer-Cartan 1-forms* $(\rho, \kappa, \zeta, \alpha, \overline{\kappa}, \overline{\zeta}, \overline{\alpha})$ *in Theorem 1.2. Then there is an isomorphism of Lie algebras between the Lie algebra generated by* $\{\partial_\rho, \partial_\kappa, \partial_\zeta, \partial_\alpha, \partial_{\overline{\kappa}}, \partial_{\overline{\zeta}}, \partial_{\overline{\alpha}}\}$ *and the Lie algebra generated by* $\{X^1, X^2, X^3, X^4, X^5, X^6, X^7\}$.

Next, when either $\boldsymbol{I}_0 \not\equiv 0$ or $\boldsymbol{V}_0 \not\equiv 0$, we may restrict considerations to either of the Zariski-open subsets $\{p \in M : \boldsymbol{I}_0(p) \neq 0\}$ or $\{p \in M : \boldsymbol{V}_0(p) \neq 0\}$, where one may perform Cartan's method of equivalence and obtain the following

Theorem 1.4. *Let* $M^5 \subset \mathbb{C}^3$ *be a local rigid* 2*-nondegenerate* $\mathscr{C}^\omega$ *constant Levi rank* 1 *hypersurface. If either* $\boldsymbol{I}_0 \neq 0$ *or* $\boldsymbol{V}_0 \neq 0$ *everywhere on* M*, the local rigid-biholomorphic equivalence problem reduces to an invariant* 5*-dimensional* $\{e\}$*-structure on* M.

In fact, once the last remaining group parameter $\mathsf{c} \in \mathbb{C}^*$ is seen to be normalizable from either:

$$\frac{1}{\mathsf{c}}\boldsymbol{I}_0 = 1 \qquad \text{or} \qquad \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{V}_0 = 1,$$

the proof is completed if one does not require to make explicit the $\{e\}$-structure on M. Because of the complexity of computations, we will not attempt to set up such an explicit $\{e\}$-structure.

From general Cartan theory, we deduce the

Corollary 1.5. *All rigid* $M^5 \subset \mathbb{C}^3$ *that are not rigidly-biholomorphic to the model* M_{LC} *satisfy*

$$\dim \mathsf{Hol}_{\mathsf{rigid}}(M) \leqslant 5.$$

Let us now briefly explain how our rigid real analytic $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$ can be equipped with a *Cartan geometry*, whatever $\boldsymbol{J}_0$, $\boldsymbol{V}_0$ are. This is analogous to the Cartan connection construced by Medori-Spiro [21, 22].

Historically, Élie Cartan introduced the notion of "*espaces généralisés*", first in the context of Riemannian geometry, then in the widest possible universe of arbitrary homogeneous spaces $X = G/H$, where G is a connected Lie group, $H \subset G$ is a closed connected Lie subgroup, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. In today's language, a Cartan-like '*generalised space*' is conceptualized as a certain $\mathfrak{g}$-valued differential 1-form ω which constitutes a *Cartan connection* on a certain H-principal bundle P over a manifold M equipped with the right action $R_h \colon g \mapsto gh$ for $h \in H$ and $g \in G$, subjected to following three key conditions:

(1) $\omega_p \colon T_pP \to \mathfrak{g}$ is an isomorphism at each point $p \in P$;

(2) for every $y \in \mathfrak{h}$, if

$$Y^+\big|_p = \frac{d}{dt}\big(p\,\mathsf{exp}(ty)\big)\big|_{t=0},$$

then $\omega(Y^+) = y$;

(3) at every $p \in P$, for every $v_p \in T_pP$, one has

$$\omega_{ph}\big(R_{h_*}(v_p)\big) = \mathsf{Ad}(h^{-1})[\omega_p(v_p)].$$

The Cartan connection for the homogeneous space $M := G/H$ with the bundle $P := G$, satisfies Maurer-Cartan structure equation:

$$0 = d\omega + \tfrac{1}{2}[\omega \wedge \omega].$$

In general, Cartan's structure equation involves the *curvature:*

$$\Omega \,:=\, d\omega + \tfrac{1}{2}[\omega \wedge \omega].$$

In our case, a general rigid real analytic $\mathfrak{C}_{2,1}$ manifold is modelled on the tube over the future light cone M_{LC}. Cartan's equivalence method realises M_{LC} as a homogenous space G^7/H^2, where $\mathfrak{g}$ is generated by the vector fields $\{\partial_\alpha, \partial_{\overline{\alpha}}, \partial_\rho, \partial_\kappa, \partial_\zeta, \partial_{\overline{\kappa}}, \partial_{\overline{\zeta}}\}$, and H^2 is the two dimensional isotropy subgroup.

From the structure equations of Theorem 1.1, it is easy to construct a $\mathfrak{g}$-valued 1-form ω satisfying the three conditions of being a Cartan connection (details will be skipped).

Theorem 1.6. *Associated to every rigid $\mathfrak{C}_{2,1}$ local $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$, there is a canonical Cartan connection modelled on the homogeneous space $G^7/H^2 = M_{LC}$.*

In continuation with these results, a further problem appears: to classify up to rigid biholomorphisms the '*submaximal*' hypersurfaces with $\mathsf{dim}\ \mathsf{Hol}_{\mathsf{rigid}}(M) = 5$ whose rigid biholomorphic group is locally transitive. Another question would be to classify under rigid biholomorphisms those rigid $M^5 \subset \mathbb{C}^3$ that have identically vanishing Pocchiola invariants $0 \equiv \boldsymbol{W}_0 \equiv \boldsymbol{J}_0$, hence which are equivalent to M_{LC}, but under a general biholomorphism, not necessarily rigid. Upcoming publications will be devoted to advances in these directions.

The remainder of the article is devoted to prove Theorem 1.1.

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2. Geometry of Levi rank 1 real hypersurfaces $M^5 \subset \mathbb{C}^3$

In appropriate affine coordinates $(z_1, z_2, w) \in \mathbb{C}^3$ with $w = u + \sqrt{-1}v$, a real-analytic ($\mathscr{C}^\omega$) real hypersurface $M^5 \subset \mathbb{C}^3$ may locally be represented as the graph of a $\mathscr{C}^\omega$ function F over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$. When F is independent of v:

$$M: \qquad u = F(z_1, z_2, \overline{z}_1, \overline{z}_2),$$

the hypersurface is called *rigid.*

Its fundamental CR-bundle:

$$T^{1,0}M \;:=\; \big(\mathbb{C} \otimes_{\mathbb{R}} TM\big) \cap T^{1,0}\mathbb{C}^3$$

is of complex rank $2 = \mathsf{CRdim} M$, as well as its conjugate $T^{0,1}M = \overline{T^{1,0}M}$.

Relevant foundational material for CR geometry focused on the local biholomorphic equivalence problem of $\mathscr{C}^\omega$ CR submanifolds $M \subset \mathbb{C}^{\mathrm{N}}$ has been set up in the memoir [28].

The *Levi forms* at various points $p \in M$ are maps measuring Lie bracket non-involutivity [28, p. 45]:

$$\begin{aligned} T_p^{1,0}M \times T_p^{1,0}M &\longrightarrow \mathbb{C} \otimes_{\mathbb{R}} T_pM && \operatorname{mod}\big(T_p^{1,0}M \oplus T_p^{0,1}M\big), \\ \big(\mathscr{M}_p,\, \mathscr{N}_p\big) &\longmapsto \sqrt{-1}\big[\mathscr{M},\, \overline{\mathscr{N}}\big]\big|_p && \operatorname{mod}\big(T_p^{1,0}M \oplus T_p^{0,1}M\big), \end{aligned}$$

where $\mathscr{M}$ and $\mathscr{N}$ are any two local sections of $T^{1,0}M$ defined near p which extend $\mathscr{M}_p = \mathscr{M}\big|_p$ and $\mathscr{N}_p = \mathscr{N}\big|_p$, the result being independent of extensions.

Levi forms are known to be biholomorphically invariant. In terms of two natural intrinsic generators for $T^{1,0}M$:

$$\mathscr{L}_1 := \frac{\partial}{\partial z_1} - \sqrt{-1}F_{z_1}\frac{\partial}{\partial v} \qquad \text{and} \qquad \mathscr{L}_2 := \frac{\partial}{\partial z_2} - \sqrt{-1}F_{z_2}\frac{\partial}{\partial v},$$

the Levi forms at all points $p \in M$ identify with the matrix-valued map:

$$\mathrm{LF}_M(p) := 2\begin{pmatrix} F_{z_1\overline{z}_1} & F_{z_2\overline{z}_1} \\ F_{z_1\overline{z}_2} & F_{z_2\overline{z}_2} \end{pmatrix}(p).$$

Throughout this article, we will make two main (invariant) assumptions. The first one is that the rank of $\mathrm{LF}_M(p)$ be constant equal to 1 at every point $p \in M$.

Since $2 = \operatorname{rank} T^{1,0}M$, this implies that there is a rank 1 *Levi kernel subbundle*:

$$K^{1,0}M \subset T^{1,0}M,$$

which is generated by the vector field:

$$\mathscr{K} := \boldsymbol{k}\mathscr{L}_1 + \mathscr{L}_2,$$

incorporating the *slant function*:

$$\boldsymbol{k} := -\frac{F_{z_2\overline{z}_1}}{F_{z_1\overline{z}_1}}.$$

Indeed, a direct check convinces that both $[\mathscr{K},\, \overline{\mathscr{L}}_1]$ and $[\mathscr{K},\, \overline{\mathscr{L}}_2]$ vanish modulo $T^{1,0}M \oplus T^{0,1}M$. The known involutivity properties of the Levi kernel subbundle $K^{1,0}M \subset T^{1,0}M$ together with its conjugate $K^{0,1}M \subset T^{0,1}M$ then read as (see [28, pp. 72-73]):

$$\begin{aligned} \big[K^{1,0}M,\, K^{1,0}M\big] &\subset K^{1,0}M, \\ \big[K^{0,1}M,\, K^{0,1}M\big] &\subset K^{0,1}M, \\ \big[K^{1,0}M,\, K^{0,1}M\big] &\subset K^{1,0}M \oplus K^{0,1}M. \end{aligned}$$

Another fundamental function will also be needed in a while:

$$\boldsymbol{P} := \frac{F_{z_1z_1\overline{z}_1}}{F_{z_1\overline{z}_1}}.$$

All this justifies the introduction of the so-called *Freeman form* ([28, p. 89]):

$$\begin{aligned} K_p^{1,0}M \times \big(T_p^{1,0}M \operatorname{mod} K_p^{1,0}M\big) &\longrightarrow T_p^{1,0}M \oplus T_p^{0,1}M && \operatorname{mod}\big(K_p^{1,0}M \oplus T_p^{0,1}M\big), \\ \big(\mathscr{K}_p,\, \mathscr{L}_p\big) &\longmapsto \big[\mathscr{K},\, \overline{\mathscr{L}}\big]\big|_p && \operatorname{mod}\big(K_p^{1,0}M \oplus T_p^{0,1}M\big), \end{aligned}$$

where $\mathscr{K}$ and $\mathscr{L}$ are any two local sections of $K^{1,0}M$ and of $T^{1,0}M$ defined near p which extend $\mathscr{K}_p = \mathscr{K}|_p$ and $\mathscr{L}_p = \mathscr{L}|_p$, the result being independent of extensions. In bases, these Freeman forms at various points $p \in M$ are simply maps $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$. They are known to be biholomorphically invariant [28].

Our second main (invariant) assumption will be that the rank of the Freeman form be maximal equal to 1 at every point $p \in M$. Such manifolds M are called 2-*nondegenerate*.

A computation:

$$
\begin{aligned}
\big[\mathscr{K},\overline{\mathscr{L}}_1\big] = \big[\boldsymbol{k}\mathscr{L}_1+\mathscr{L}_2,\overline{\mathscr{L}}_1\big] &= -\overline{\mathscr{L}}_1(\boldsymbol{k})\mathscr{L}_1 + \underline{\boldsymbol{k}\,\big[\mathscr{L}_1,\overline{\mathscr{L}}_1\big]+\big[\mathscr{L}_2,\overline{\mathscr{L}}_1\big]}_{\circ} \\
&= -\overline{\mathscr{L}}_1(\boldsymbol{k})\mathscr{L}_1
\end{aligned}
\tag{2.1}
$$

shows that

$$
M \text{ is 2-nondegenerate at } p\in M \qquad\Longleftrightarrow\qquad \overline{\mathscr{L}}_1(\boldsymbol{k})(p)\neq 0.
$$

2.2. The initial Darboux-Cartan structure. The differential 1-form

$$
\rho_0 = dv + \sqrt{-1}F_{z_1}\,dz^1 + \sqrt{-1}F_{z_2}\,dz^2 - \sqrt{-1}F_{\bar z_1}\,d\bar z^1 - \sqrt{-1}F_{\bar z_2}\,d\bar z^2
$$

has kernel

$$
\ker\rho_0 = \{\rho_0=0\} = T^{1,0}M\oplus T^{0,1}M.
$$

If M is not Levi-flat, after a suitable change of coordinates in the (z_1,z_2)-space, we may assume without loss of generality that:

$$
\rho_0\big(\sqrt{-1}[\mathscr{L}_1,\overline{\mathscr{L}}_2]\big) = 2F_{z_1\bar z_1}\neq 0
$$

everywhere on M, and hence the vector field

$$
\mathscr{T} := \sqrt{-1}[\mathscr{L}_1,\overline{\mathscr{L}}_1] = 2F_{z_1,\bar z_1}\frac{\partial}{\partial v} := \ell\frac{\partial}{\partial v}
$$

vanishes nowhere on M.

In the rigid case, a direct calculation shows that

$$
\begin{aligned}
\mathscr{L}_1(\boldsymbol{k}) &= -\frac{-F_{z_1,\bar z_1}F_{z_2\bar z_1 z_1}+F_{z_2\bar z_1}F_{z_1\bar z_1 z_1}}{(F_{z_1\bar z_1})^2},\\
\overline{\mathscr{L}}_1(\boldsymbol{k}) &= \frac{-F_{z_1\bar z_1}F_{z_2\bar z_1\bar z_1}+F_{z_2\bar z_1}F_{z_1\bar z_1\bar z_1}}{(F_{z_1\bar z_1})^2},\\
\mathscr{T}(\boldsymbol{k}) &= 0.
\end{aligned}
$$

Moreover, we will invoke the following:

Lemma 2.3. [See Pocchiola [30] or Foo-Merker [8]] *The following* 3 *functional identities hold on* M*:*

$$
\begin{aligned}
&\mathscr{K}(\bar{\boldsymbol{k}})\equiv 0,\\
&\mathscr{K}(\boldsymbol{P})\equiv -\boldsymbol{P}\mathscr{L}_1(\boldsymbol{k}) - \mathscr{L}_1(\mathscr{L}_1(\boldsymbol{k})),\\
&\mathscr{K}(\overline{\boldsymbol{P}})\equiv -\boldsymbol{P}\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1(\mathscr{L}_1(\boldsymbol{k})).
\end{aligned}
$$

According to Pocchiola [30, p. 37], there are 10 Lie bracket identities

$$
\begin{aligned}
&[\mathscr{T},\mathscr{L}_1]\equiv -\boldsymbol{P}\mathscr{T}, && [\mathscr{L}_1,\overline{\mathscr{L}}_1]\equiv\sqrt{-1}\mathscr{T},\\
&[\mathscr{T},\mathscr{K}]\equiv \mathscr{L}_1(\boldsymbol{k})\mathscr{T}+0, && [\mathscr{L}_1,\overline{\mathscr{K}}]\equiv \mathscr{L}_1(\bar{\boldsymbol{k}})\overline{\mathscr{L}}_1,\\
&[\mathscr{T},\overline{\mathscr{L}}_1]\equiv -\overline{\boldsymbol{P}}\mathscr{T}, && [\mathscr{K},\overline{\mathscr{L}}_1]\equiv -\overline{\mathscr{L}}_1(\boldsymbol{k})\mathscr{L}_1,\\
&[\mathscr{T},\overline{\mathscr{K}}]\equiv \overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})\mathscr{T}+0, && [\mathscr{K},\overline{\mathscr{K}}]\equiv 0,\\
&[\mathscr{L}_1,\mathscr{K}]\equiv \mathscr{L}_1(\boldsymbol{k})\mathscr{L}_1, && [\overline{\mathscr{L}}_1,\overline{\mathscr{K}}]\equiv \overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})\overline{\mathscr{L}}_1,
\end{aligned}
$$

where the "+0" is deliberately added to show the difference from the general case. The following 1-forms

$$\begin{aligned}
\rho_0 &= \frac{1}{\ell}\big(dv - A^1 dz_1 - A^2 dz_2 - \bar{A}^1 d\bar{z}_1 - \bar{A}^2 d\bar{z}_2\big),\\
\kappa_0 &= dz_1 - \boldsymbol{k} dz_2,\\
\zeta_0 &= dz_2,\\
\bar{\kappa}_0 &= d\bar{z}_1 - \bar{\boldsymbol{k}} d\bar{z}_2,\\
\bar{\zeta}_0 &= d\bar{z}_2,
\end{aligned}$$

are, by a simple computation, dual to the corresponding vector fields $\mathscr{T}$, $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}}_1$, $\overline{\mathscr{K}}$. In terms of these new vector fields and 1-forms, the exterior differential of any $\mathscr{C}^\omega$ function $G = G(z_1, z_2, \overline{z}_1, \overline{z}_2, v)$ rewrites simply as

$$dG = \mathscr{T}(G)\,\rho_0 + \mathscr{L}_1(G)\,\kappa_0 + \mathscr{K}(G)\,\zeta_0 + \overline{\mathscr{L}}_1(G)\,\overline{\kappa}_0 + \overline{\mathscr{K}}(G)\,\overline{\zeta}_0. \tag{2.4}$$

Using the Lie-Cartan formula which states that for any smooth vector fields X, Y and any differential 1-form ω, one has

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$

the initial Darboux-Cartan structure equations are therefore obtained

$$\begin{aligned}
d\rho_0 &= \boldsymbol{P}\,\rho_0\wedge\kappa_0 - \mathscr{L}_1(\boldsymbol{k})\,\rho_0\wedge\zeta_0 + \overline{\boldsymbol{P}}\,\rho_0\wedge\bar{\kappa}_0 - \overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})\,\rho_0\wedge\bar{\zeta}_0 + \sqrt{-1}\kappa_0\wedge\bar{\kappa}_0,\\
d\kappa_0 &= -\mathscr{L}_1(\boldsymbol{k})\,\kappa_0\wedge\zeta_0 + \overline{\mathscr{L}}_1(\boldsymbol{k})\,\zeta_0\wedge\bar{\kappa}_0,\\
d\zeta_0 &= 0.
\end{aligned}$$

Here, conjugate equations for $d\overline{\kappa}_0$ and for $d\overline{\zeta}_0$ are not written, as they can be immediately deduced.

By anticipation, let us state that Cartan's method will force us to replace the three independent 1-forms $\{\rho_0, \kappa_0, \zeta_0\}$, first by $\{\rho_0, \kappa_0, \hat{\zeta}_0\}$, next by $\{\rho_0, \kappa_0, \zeta_0'\}$, and that we will have to calculate more complicated initial structure equations.

3. Initial G-structure for rigid equivalences of rigid real hypersurfaces

Our objective is to study the equivalence problem of rigid hypersurfaces under rigid biholomorphic transformations.

Définition 3.1. Two local $\mathscr{C}^\omega$ rigid real hypersurfaces $M^5 \subset \mathbb{C}^3$ and $M'^5 \subset \mathbb{C}'^3$ are said to be *rigidly equivalent* if there exists a (local) biholomorphic map of the form:

$$\varphi\colon\quad (z_1, z_2, w) \;\longmapsto\; \big(f(z_1,z_2), g(z_1,z_2), aw + h(z_1,z_2)\big) \;=:\; (z_1', z_2', w'),$$

sending M to M', where $a \in \mathbb{R}^\times$ and f, g, h are holomorphic of (z_1, z_2) only.

The interest, advocated by Stanton and by Isaev, is that *rigid biholomorphisms preserve rigidity*. Indeed, starting from the target rigid hypersurface

$$\frac{w' + \bar{w}'}{2} - F'(z_1', z_2', \bar{z}_1', \bar{z}_2') = 0,$$

the pullback by φ again has rigid defining equation

$$0 = \frac{w+\bar{w}}{2} + \frac{1}{a}\Big(\frac{1}{2}h(z_1,z_2) + \frac{1}{2}\bar{h}(\bar{z}_1,\bar{z}_2) - F'\big(f(z_1,z_2), g(z_1,z_2), \bar{f}(\bar{z}_1,\bar{z}_2), \bar{g}(\bar{z}_1\bar{z}_2)\big)\Big).$$

Since φ is holomorphic, its differential $\varphi_* : \mathbb{C}T\mathbb{C}^3 \to \mathbb{C}T\mathbb{C}'^3$ stabilises the holomorphic $(1,0)$ and the anti-holomorphic $(0,1)$ vector bundles

$$\begin{aligned} \varphi_* T^{1,0}M &\subseteq T^{1,0}M', \\ \varphi_* T^{0,1}M &\subseteq T^{0,1}M'. \end{aligned}$$

Furthermore, by the invariance of the Freeman forms, φ_* also respects the Levi kernel bundles

$$\varphi_* K^{1,0}M \subset K^{1,0}M'.$$

Consequently, there exist functions f', c', e' on M' such that

$$\begin{aligned} \varphi_*(\mathscr{K}) &= \mathsf{f}'\mathscr{K}', \\ \varphi_*(\mathscr{L}_1) &= \mathsf{c}'\mathscr{L}_1' + \mathsf{e}'\mathscr{K}'. \end{aligned} \tag{3.2}$$

Next, if $R'(z_1', z_2', \bar{z}_1', \bar{z}_2', v')$ is any $\mathscr{C}^\omega$ function on M', then by definition of the push-forward of a vector field, with $\mathscr{T} = \ell\partial_v$ and $\mathscr{T}' = \ell'\partial_{v'}$, we have

$$\begin{aligned} (\varphi_*\mathscr{T})\Big(R'(z_1', z_2', \bar{z}_1', \bar{z}_2', v')\Big) &= \mathscr{T}(R' \circ \varphi) \\ &= \ell\frac{\partial}{\partial v}\Big(R'\big(f(z_1,z_2), g(z_1,z_2), \overline{f}(\bar{z}_1,\bar{z}_2), \overline{g}(\bar{z}_1,\bar{z}_2), av + \mathsf{Im}\, h(z_1,z_2)\big)\Big) \\ &= a\ell\frac{\partial R'}{\partial v'}\circ\varphi = \frac{a\,\ell}{\ell'\circ\varphi}\bigg(\ell'\circ\varphi\,\frac{\partial R'}{\partial v'}\circ\varphi\bigg) = \frac{a\,\ell}{\ell'\circ\varphi}(\mathscr{T}'R')\circ\varphi. \end{aligned}$$

Hence, there exists a real-valued function a' nowhere vanishing on M' such that

$$\varphi_*\mathscr{T} = \mathsf{a}'\mathscr{T}'.$$

In fact, this function is determined as $\mathsf{a}' = \mathsf{c}'\bar{\mathsf{c}}'$, since by using (3.2), (2.1), we see that

$$\begin{aligned} \mathsf{a}'\mathscr{T}' = \varphi_*\mathscr{T} &= \varphi_*\big(\sqrt{-1}[\mathscr{L}_1, \overline{\mathscr{L}}_1]\big) \\ &= \sqrt{-1}[\varphi_*\mathscr{L}_1, \varphi_*\overline{\mathscr{L}}_1] \\ &= \mathsf{c}'\bar{\mathsf{c}}'\sqrt{-1}[\mathscr{L}_1', \overline{\mathscr{L}}_1'] \quad \mod T^{1,0}M' \oplus T^{0,1}M'. \end{aligned}$$

Summarising, we therefore have the following matrix

$$\varphi_*\begin{pmatrix} \mathscr{T} \\ \mathscr{L}_1 \\ \mathscr{K} \\ \overline{\mathscr{L}}_1 \\ \overline{\mathscr{K}} \end{pmatrix} = \begin{pmatrix} \mathsf{c}'\bar{\mathsf{c}}' & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}' & \mathsf{e}' & 0 & 0 \\ 0 & 0 & \mathsf{f}' & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathsf{c}}' & \bar{\mathsf{e}}' \\ 0 & 0 & 0 & 0 & \bar{\mathsf{f}}' \end{pmatrix}\begin{pmatrix} \mathscr{T}' \\ \mathscr{L}_1' \\ \mathscr{K}' \\ \overline{\mathscr{L}}_1' \\ \overline{\mathscr{K}}' \end{pmatrix}.$$

Transposing the matrix, we obtain the pullback formula for the two coframes

$$\varphi^*\begin{pmatrix} \rho_0' \\ \kappa_0' \\ \zeta_0' \\ \bar{\kappa}_0' \\ \bar{\zeta}_0' \end{pmatrix} = \begin{pmatrix} \mathsf{c}'\bar{\mathsf{c}}' & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}' & 0 & 0 & 0 \\ 0 & \mathsf{e}' & \mathsf{f}' & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathsf{c}}' & 0 \\ 0 & 0 & 0 & \bar{\mathsf{e}}' & \bar{\mathsf{f}}' \end{pmatrix}\begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \bar{\kappa}_0 \\ \bar{\zeta}_0 \end{pmatrix}.$$

In conclusion, for our rigid equivalence problem, the initial G-structure is constituted by the following 5 by 5 matrices

$$\begin{pmatrix} \mathsf{c}\bar{\mathsf{c}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c} & 0 & 0 & 0 \\ 0 & \mathsf{e} & \mathsf{f} & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathsf{c}} & 0 \\ 0 & 0 & 0 & \bar{\mathsf{e}} & \bar{\mathsf{f}} \end{pmatrix},$$

with the free complex variables

$$\mathsf{c},\mathsf{f}\in\mathbb{C}\backslash\{0\},\qquad\qquad \mathsf{e}\in\mathbb{C}.$$

Henceforth, we will forget about the conjugated 1-form, and the initial G-structure that we need is represented by the *lifted coframe:*

$$\omega := \begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} := \begin{pmatrix}\mathsf{c}\bar{\mathsf{c}} & 0 & 0\\ 0 & \mathsf{c} & 0\\ 0 & \mathsf{e} & \mathsf{f}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0\end{pmatrix} =: \mathsf{g}\omega_0.$$

In the next few sections, we will perform reductions of this G-structure by making suitable changes to the horizontal coframe

$$(\rho_0,\zeta_0,\kappa_0)\longrightarrow\big(\rho_0,\,\overline{\mathscr{L}}_1(\boldsymbol{k})\zeta_0,\,\kappa_0\big)\longrightarrow\big(\rho_0,\,\overline{\mathscr{L}}_1(\boldsymbol{k})\zeta_0+\boldsymbol{B}\kappa_0,\,\kappa_0\big),$$

where $\boldsymbol{B}$ is a certain function, corresponding to the group reductions

$$\begin{pmatrix}\mathsf{c}\bar{\mathsf{c}} & 0 & 0\\ 0 & \mathsf{c} & 0\\ 0 & \mathsf{e} & \mathsf{f}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathsf{c}\bar{\mathsf{c}} & 0 & 0\\ 0 & \mathsf{c} & 0\\ 0 & \mathsf{e} & \frac{\mathsf{c}}{\bar{\mathsf{c}}}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathsf{c}\bar{\mathsf{c}} & 0 & 0\\ 0 & \mathsf{c} & 0\\ 0 & 0 & \frac{\mathsf{c}}{\bar{\mathsf{c}}}\end{pmatrix}.$$

4. Cartan process: first loop

In the exterior derivative of matrix group formula

$$d\omega=(d\mathsf{g})\mathsf{g}^{-1}\omega+\mathsf{g}d\omega_0,$$

the Maurer-Cartan matrix is

$$(d\mathsf{g})\mathsf{g}^{-1}=\begin{pmatrix}\alpha+\bar\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & \delta & \varepsilon\end{pmatrix},$$

where

$$\alpha:=\frac{d\mathsf{c}}{\mathsf{c}},\qquad\qquad \delta:=\frac{d\mathsf{e}}{\mathsf{c}}-\frac{\mathsf{e}}{\mathsf{c}}\frac{d\mathsf{f}}{\mathsf{f}},\qquad\qquad \varepsilon:=\frac{d\mathsf{f}}{\mathsf{f}}.$$

A direct computation gives

$$\begin{aligned}
d\rho &= \alpha\wedge\rho+\bar\alpha\wedge\rho+\left(\frac{\boldsymbol{P}}{\mathsf{c}}+\frac{\mathsf{e}\mathscr{L}_1(\boldsymbol{k})}{\mathsf{c}\mathsf{f}}\right)\rho\wedge\kappa+\left(\frac{\bar{\boldsymbol{P}}}{\bar{\mathsf{c}}}+\frac{\bar{\mathsf{e}}\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{\bar{\mathsf{c}}\bar{\mathsf{f}}}\right)\rho\wedge\bar\kappa\\
&\quad+\left(\frac{-\mathscr{L}_1(\boldsymbol{k})}{\mathsf{f}}\right)\rho\wedge\zeta+\left(\frac{-\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{\bar{\mathsf{f}}}\right)\rho\wedge\bar\zeta+\sqrt{-1}\kappa\wedge\bar\kappa,\\
d\kappa &= \alpha\wedge\kappa+\left(\frac{-\mathscr{L}_1(\kappa)}{\mathsf{f}}\right)\kappa\wedge\zeta+\left(-\frac{\mathsf{e}\overline{\mathscr{L}}_1(\boldsymbol{k})}{\bar{\mathsf{c}}\mathsf{f}}\right)\kappa\wedge\bar\kappa+\left(\frac{\mathsf{c}\overline{\mathscr{L}}_1(\boldsymbol{k})}{\bar{\mathsf{c}}\mathsf{f}}\right)\zeta\wedge\bar\kappa,\\
d\zeta &= \delta\wedge\kappa+\varepsilon\wedge\zeta+\left(\frac{-\mathsf{e}\mathscr{L}_1(\boldsymbol{k})}{\mathsf{c}\mathsf{f}}\right)\kappa\wedge\zeta+\left(\frac{-\mathsf{e}^2\overline{\mathscr{L}}_1(\boldsymbol{k})}{\mathsf{c}\bar{\mathsf{c}}\mathsf{f}}\right)\kappa\wedge\bar\kappa\\
&\quad+\left(\frac{\mathsf{e}\mathscr{L}_1(\boldsymbol{k})}{\bar{\mathsf{c}}\mathsf{f}}\right)\zeta\wedge\bar\kappa.
\end{aligned}\tag{4.1}$$

Next, we proceed with the absorption by introducing 3×5 indeterminates

$$\begin{aligned}
\alpha &= \boldsymbol{\alpha}-x_\rho\rho-x_\kappa\kappa-x_\zeta\zeta-x_{\bar\kappa}\bar\kappa-x_{\bar\zeta}\bar\zeta,\\
\delta &= \boldsymbol{\delta}-y_\rho\rho-y_\kappa\kappa-y_\zeta\zeta-y_{\bar\kappa}\bar\kappa-y_{\bar\zeta}\bar\zeta,\\
\varepsilon &= \boldsymbol{\varepsilon}-z_\rho\rho-z_\kappa\kappa-z_\zeta\zeta-z_{\bar\kappa}\bar\kappa-z_{\bar\zeta}\bar\zeta.
\end{aligned}$$

By solving a system of linear equations in order to eliminate as many torsion coefficients as possible in (4.1), we find values of these indeterminates to arrange that

$$\begin{aligned} d\rho &= (\boldsymbol{\alpha} + \overline{\boldsymbol{\alpha}}) \wedge \rho + \sqrt{-1}\kappa \wedge \bar{\kappa}, \\ d\kappa &= \boldsymbol{\alpha} \wedge \kappa + \frac{\mathsf{c}\overline{\mathscr{L}}_1(\boldsymbol{k})}{\bar{\mathsf{c}}\mathsf{f}}\, \zeta \wedge \bar{\kappa}, \\ d\zeta &= \boldsymbol{\delta} \wedge \kappa + \boldsymbol{\varepsilon} \wedge \zeta. \end{aligned}$$

Notice that the nowhere vanishing function appearing in $d\kappa$

$$\frac{\mathsf{c}\overline{\mathscr{L}}_1(\boldsymbol{k})}{\bar{\mathsf{c}}\mathsf{f}}$$

is an essential (not absorbable) torsion coefficient, so from general Cartan theory, it is invariant under equivalences, hence it may be normalised to 1 by setting

$$\mathsf{f} := \frac{\mathsf{c}\overline{\mathscr{L}}_1(\boldsymbol{k})}{\bar{\mathsf{c}}}.$$

5. Cartan process: second loop

This normalisation of f conducts us to change our base coframe by introducing

$$\hat{\zeta}_0 := \overline{\mathscr{L}}_1(\boldsymbol{k})\, \zeta_0,$$

so that the new G-structure (without f) and new lifted coframe become

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathsf{c}\bar{\mathsf{c}} & 0 & 0 \\ 0 & \mathsf{c} & 0 \\ 0 & \mathsf{e} & \frac{\mathsf{c}}{\bar{\mathsf{c}}} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \hat{\zeta}_0 \end{pmatrix}.$$

In the new coframe $\{\rho_0, \kappa_0, \hat{\zeta}_0\}$, the exterior differential (2.4) of any $\mathscr{C}^\omega$ function G on M becomes

$$dG = \mathscr{T}(G)\, \rho_0 + \mathscr{L}_1(G)\, \kappa_0 + \frac{1}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\, \mathscr{K}(G)\, \hat{\zeta}_0 + \overline{\mathscr{L}}_1(G)\, \overline{\kappa}_0 + \frac{1}{\mathscr{L}_1(\bar{\boldsymbol{k}})}\, \overline{\mathscr{K}}(G)\, \bar{\hat{\zeta}}_0.$$

Since both $\boldsymbol{k}$ and $\mathscr{L}_1(\boldsymbol{k})$ are independent of v, we have

$$\mathscr{T}(\boldsymbol{k}) \equiv 0, \qquad \mathscr{T}(\mathscr{L}_1(\boldsymbol{k})) \equiv 0.$$

Borrowing equation (5.5) of Foo-Merker [8] and its proof, or proceeding directly, the reader will see that the new Darboux-Cartan structure equations become

$$\begin{aligned} d\rho_0 &= \boldsymbol{P}\, \rho_0 \wedge \kappa_0 - \frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\, \rho_0 \wedge \hat{\zeta}_0 + \overline{\boldsymbol{P}}\, \rho_0 \wedge \bar{\kappa}_0 - \frac{\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{\mathscr{L}_1(\bar{\boldsymbol{k}})}\, \rho_0 \wedge \overline{\hat{\zeta}_0} + \sqrt{-1}\kappa_0 \wedge \bar{\kappa}_0, \\ d\kappa_0 &= -\frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\, \kappa_0 \wedge \hat{\zeta}_0 + \hat{\zeta}_0 \wedge \bar{\kappa}_0, \\ d\hat{\zeta}_0 &= \frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\, \kappa_0 \wedge \hat{\zeta}_0 - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\, \hat{\zeta}_0 \wedge \overline{\kappa}_0 + \frac{\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{\mathscr{L}_1(\bar{\boldsymbol{k}})}\, \hat{\zeta}_0 \wedge \overline{\hat{\zeta}_0}. \end{aligned} \tag{5.1}$$

Moreover, the Maurer-Cartan matrix is

$$(d\mathsf{g})\mathsf{g}^{-1} = \begin{pmatrix} \alpha + \bar{\alpha} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \delta & \alpha - \bar{\alpha} \end{pmatrix},$$

with the 1-forms

$$\alpha := \frac{d\mathsf{c}}{\mathsf{c}}, \qquad \delta := \frac{d\mathsf{e}}{\mathsf{c}} - \frac{\mathsf{e}}{\mathsf{c}}\left(\frac{d\mathsf{c}}{\mathsf{c}} - \frac{d\bar{\mathsf{c}}}{\bar{\mathsf{c}}}\right).$$

After computations, we obtain

$$d\rho = (\alpha+\bar\alpha)\wedge\rho + \left(\frac{P}{\mathsf{c}} + \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}}_1(k)}\frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}^2}\right)\rho\wedge\kappa + \left(-\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}}_1(k)}\frac{\bar{\mathsf{c}}}{\mathsf{c}}\right)\rho\wedge\zeta$$
$$+\left(\frac{\bar P}{\bar{\mathsf{c}}} + \frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2}\right)\rho\wedge\bar\kappa + \left(-\frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\mathsf{c}}{\bar{\mathsf{c}}}\right)\rho\wedge\bar\zeta + \sqrt{-1}\kappa\wedge\bar\kappa,$$
$$d\kappa = \alpha\wedge\kappa + \left(-\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}}_1(k)}\frac{\bar{\mathsf{c}}}{\mathsf{c}}\right)\kappa\wedge\zeta - \frac{\mathsf{e}}{\mathsf{c}}\,\kappa\wedge\bar\kappa + \zeta\wedge\bar\kappa,$$
$$d\zeta = \delta\wedge\kappa + (\alpha-\bar\alpha)\wedge\zeta + \left(-\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}}_1(k)}\frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}^2} + \frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)}\frac{1}{\mathsf{c}}\right)\kappa\wedge\zeta$$
$$+\left(-\frac{\mathsf{e}^2}{\mathsf{c}^2} + \frac{\overline{\mathscr{L}}(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\mathsf{e}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2} + \frac{\overline{\mathscr{L}}_1(\mathscr{L}_1(k))}{\overline{\mathscr{L}}_1(k)}\frac{\mathsf{e}}{\mathsf{c}\bar{\mathsf{c}}}\right)\kappa\wedge\bar\kappa + \left(\frac{\mathsf{e}}{\mathsf{c}} - \frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2} - \frac{\overline{\mathscr{L}}_1(\mathscr{L}_1(k))}{\overline{\mathscr{L}}_1(k)}\frac{1}{\bar{\mathsf{c}}}\right)\zeta\wedge\bar\kappa$$
$$-\frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\mathsf{e}}{\bar{\mathsf{c}}}\,\kappa\wedge\bar\zeta + \frac{\mathsf{c}\overline{\mathscr{L}}_1(\bar k)}{\bar{\mathsf{c}}\mathscr{L}_1(\bar k)}\,\zeta\wedge\bar\zeta.$$

As before, we proceed with the absorption by setting

$$\alpha =: \boldsymbol{\alpha} - x_\rho\rho - x_\kappa\kappa - x_\zeta\zeta - x_{\bar\kappa}\bar\kappa - x_{\bar\zeta}\bar\zeta,$$
$$\delta =: \boldsymbol{\delta} - y_\rho\rho - y_\kappa\kappa - y_\zeta\zeta - y_{\bar\kappa}\bar\kappa - y_{\bar\zeta}\bar\zeta.$$

By examining all the absorption equations which would conduct to some essential torsions (appropriate linear combinations of torsion coefficients), we come to three key equations

$$x_{\bar\kappa} + \overline{x_\kappa} = -\frac{\bar P}{\bar{\mathsf{c}}} - \frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2},$$
$$x_{\bar\kappa} = \frac{\mathsf{e}}{\mathsf{c}},$$
$$x_{\bar\kappa} - \overline{x_\kappa} = -\frac{\mathsf{e}}{\mathsf{c}} + \frac{\overline{\mathscr{L}}_1(\bar k)}{\mathscr{L}_1(\bar k)}\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2} + \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)}\frac{1}{\bar{\mathsf{c}}}.$$

After elimination of the two indeterminates x_κ, $x_{\overline{\kappa}}$ on the left, we receive on the right an essential torsion combination which, when set equal to zero, conducts us to *normalize* the group parameter

$$\mathsf{e} := \frac{\mathsf{c}}{\bar{\mathsf{c}}}\left(-\frac{1}{3}\bar P + \frac{1}{3}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)}\right).$$

We would like to remark that in [8], a normalisation is also done during the second loop of the Cartan process, not of e, but of a certain group parameter b (absent or equal to 0 in the rigid context), namely

$$\mathsf{b} := -\sqrt{-1}\bar{\mathsf{c}}\mathsf{e} + \frac{\sqrt{-1}}{3}\mathsf{c}\left(\frac{\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(k)}{\overline{\mathscr{L}}_1(k)} - \bar P\right),$$

and when $\mathsf{b} = 0$ (rigidity assumption), the above normalization for e pops up again!

Before proceeding to the final loop of the Cartan process, let us set

$$B := -\frac{1}{3}\bar P + \frac{1}{3}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)},$$

6. Final loop

We now make a final change of base coframe by setting

$$\zeta_0' = \hat\zeta_0 + B\kappa_0,$$

so that the reduced G-structure and lifted coframe become

$$\begin{pmatrix}\rho\\ \kappa\\ \zeta\end{pmatrix} = \begin{pmatrix}\mathsf{c}\overline{\mathsf{c}} & 0 & 0\\ 0 & \mathsf{c} & 0\\ 0 & 0 & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\end{pmatrix}\begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0'\end{pmatrix}. \tag{6.1}$$

At this stage, the computation of the Darboux-Cartan structure of $\{\rho_0, \kappa_0, \zeta_0'\}$ requires some work. The exterior differential of any function G independent of c and $\overline{\mathsf{c}}$ becomes

$$\begin{aligned} dG &= \mathscr{T}(G)\rho_0 + \left(\mathscr{L}_1(G) - \boldsymbol{B}\frac{\mathscr{K}(G)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right)\kappa_0 + \frac{\mathscr{K}(G)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\zeta_0' \\ &\quad + \left(\overline{\mathscr{L}}_1(G) - \overline{\boldsymbol{B}}\frac{\overline{\mathscr{K}}(G)}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\right)\overline{\kappa}_0 + \frac{\overline{\mathscr{K}}(G)}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\overline{\zeta}_0' \\ &:= \partial_{\rho_0}(G)\,\rho_0 + \partial_{\kappa_0}(G)\,\kappa_0 + \partial_{\zeta_0'}(G)\,\zeta_0' + \partial_{\overline{\kappa}_0}(G)\,\overline{\kappa}_0 + \partial_{\overline{\zeta}_0'}(G)\,\overline{\zeta}_0'. \end{aligned} \tag{6.2}$$

After replacement of $\hat{\zeta}_0 = \zeta_0' - \boldsymbol{B}\kappa_0$ in $d\rho_0$ from (5.1), we may re-express

$$\begin{aligned} d\rho_0 &= \left(\boldsymbol{P} - \frac{\overline{\boldsymbol{P}}\mathscr{L}_1(\boldsymbol{k})}{3\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))\mathscr{L}_1(\boldsymbol{k})}{3\overline{\mathscr{L}}_1(\boldsymbol{k})^2}\right)\rho_0\wedge\kappa_0 - \frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\rho_0\wedge\zeta_0' \\ &\quad + \left(\overline{\boldsymbol{P}} - \frac{\boldsymbol{P}\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{3\mathscr{L}_1(\overline{\boldsymbol{k}})} + \frac{\mathscr{L}_1(\mathscr{L}_1(\overline{\boldsymbol{k}}))\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{3\mathscr{L}_1(\overline{\boldsymbol{k}})^2}\right)\rho_0\wedge\overline{\kappa}_0 - \frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\,\rho_0\wedge\overline{\zeta}_0' + \sqrt{-1}\kappa_0\wedge\overline{\kappa}_0 \\ &=: \boldsymbol{R}_1\,\rho_0\wedge\kappa_0 + \boldsymbol{R}_2\,\rho_0\wedge\zeta_0' + \overline{\boldsymbol{R}}_1\,\rho_0\wedge\overline{\kappa}_0 + \overline{\boldsymbol{R}}_2\,\rho_0\wedge\overline{\zeta}_0' + \sqrt{-1}\kappa_0\wedge\overline{\kappa}_0, \end{aligned}$$

Notice that two abbreviated quantities $\boldsymbol{R}_1, \boldsymbol{R}_2$ have been implicitly introduced. Similarly

$$\begin{aligned} d\kappa_0 &= -\frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\kappa_0\wedge\zeta_0' + \left(\frac{\overline{\boldsymbol{P}}}{3} - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})}\right)\kappa_0\wedge\overline{\kappa}_0 + \zeta_0'\wedge\overline{\kappa}_0 \\ &=: \boldsymbol{K}_5\,\kappa_0\wedge\zeta_0' + \boldsymbol{K}_6\,\kappa_0\wedge\overline{\kappa}_0 + \zeta_0'\wedge\overline{\kappa}_0, \end{aligned} \tag{6.3}$$

The computation of $d\zeta_0'$ starts as

$$d\zeta_0' = d\hat{\zeta}_0 + d\boldsymbol{B}\wedge\kappa_0 + \boldsymbol{B}\,d\kappa_0.$$

The first term is also treated by a plain replacement of $\hat{\zeta}_0 = \zeta_0' - \boldsymbol{B}\kappa_0$ in $d\zeta_0$ from (5.1):

$$\begin{aligned} d\hat{\zeta}_0 &= \frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\kappa_0\wedge\zeta_0' - \boldsymbol{B}\frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\,\kappa_0\wedge\overline{\zeta}_0' - \left(\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \overline{\boldsymbol{B}}\frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\right)\zeta_0'\wedge\overline{\kappa}_0 \\ &\quad + \left(\boldsymbol{B}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \boldsymbol{B}\overline{\boldsymbol{B}}\frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\right)\kappa_0\wedge\overline{\kappa}_0 + \frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\,\zeta_0'\wedge\overline{\zeta}_0', \end{aligned}$$

as well as the third term

$$\boldsymbol{B}\,d\kappa_0 = -\boldsymbol{B}\,\frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\kappa_0\wedge\zeta_0' - \boldsymbol{B}^2\,\kappa_0\wedge\overline{\kappa}_0 + \boldsymbol{B}\,\zeta_0'\wedge\overline{\kappa}_0.$$

The second term $d\boldsymbol{B}\wedge\kappa_0$ is more delicate. One not only needs (6.2) applied to $G := \boldsymbol{B}$ observing that $\mathscr{T}(\boldsymbol{B}) \equiv 0$ by the rigidity assumption, but also, one needs the following key relation coming from the Levi rank 1 assumption (use Lemma 2.3 or borrow Assertion 7.4 of Foo-Merker [8])

$$\overline{\mathscr{K}}(\boldsymbol{B}) = -\boldsymbol{B}\overline{\mathscr{L}}_1(\overline{\boldsymbol{k}}).$$

Summing carefully, observing that $\mathscr{L}_1(\cdot)$ and $\overline{\mathscr{L}}_1(\cdot)$ commute on functions independent of v, and reorganizing patiently conducts to

$$
\begin{aligned}
d\zeta_0' = \bigg(&\frac{\overline{\boldsymbol{P}}\mathscr{L}_1(\boldsymbol{k})}{3\overline{\mathscr{L}}_1(\boldsymbol{k})} - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))\mathscr{L}_1(\boldsymbol{k})}{3\overline{\mathscr{L}}_1(\boldsymbol{k})^2} + \frac{2\mathscr{L}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})} - \frac{\boldsymbol{P}}{3} \\
&\qquad - \frac{\mathscr{K}(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k})))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})^2} + \frac{\mathscr{K}(\overline{\mathscr{L}}_1(\boldsymbol{k}))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})^3}\bigg)\,\kappa_0\wedge\zeta_0' \\
&+ \bigg(\frac{-\overline{\boldsymbol{P}}^2}{9} - \frac{\overline{\boldsymbol{P}}\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{9\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{5\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))^2}{9\overline{\mathscr{L}}_1(\boldsymbol{k})^2} - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k})))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{\overline{\mathscr{L}}_1(\overline{\boldsymbol{P}})}{3}\bigg)\kappa_0\wedge\overline{\kappa}_0 \\
&+ \bigg(\frac{-\overline{\boldsymbol{P}}}{3} - \frac{2\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\boldsymbol{k}))}{3\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{\boldsymbol{P}\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{3\mathscr{L}_1(\bar{\boldsymbol{k}})} - \frac{\mathscr{L}_1(\mathscr{L}_1(\bar{\boldsymbol{k}}))\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{3\mathscr{L}_1(\bar{\boldsymbol{k}})^2}\bigg)\zeta_0'\wedge\overline{\kappa}_0 + \frac{\overline{\mathscr{L}}_1(\bar{\boldsymbol{k}})}{\mathscr{L}_1(\bar{\boldsymbol{k}})}\,\zeta_0'\wedge\overline{\zeta}_0' \\
=:{} & \boldsymbol{Z}_5\;\kappa_0\wedge\zeta_0' + \boldsymbol{Z}_6\;\kappa_0\wedge\bar{\kappa}_0 + \boldsymbol{Z}_8\;\zeta_0'\wedge\bar{\kappa}_0 + \boldsymbol{Z}_9\;\zeta_0'\wedge\bar{\zeta}_0'.
\end{aligned}
\tag{6.4}
$$

Again, notice that abbreviated quantities $\boldsymbol{Z}_5, \boldsymbol{Z}_6, \boldsymbol{Z}_8, \boldsymbol{Z}_9$ are introduced.

Thanks to this preliminary, the new lifted 1-forms ρ, κ, ζ from (6.1) have differentials

$$
\begin{aligned}
d\rho &= (\alpha+\bar{\alpha})\wedge\rho + \frac{1}{\mathsf{c}}\boldsymbol{R}_1\;\rho\wedge\kappa + \frac{\bar{\mathsf{c}}}{\mathsf{c}}\boldsymbol{R}_2\;\rho\wedge\zeta + \frac{1}{\bar{\mathsf{c}}}\overline{\boldsymbol{R}}_1\;\rho\wedge\bar{\kappa} + \frac{\mathsf{c}}{\bar{\mathsf{c}}}\;\overline{\boldsymbol{R}}_2\;\rho\wedge\bar{\zeta} + \sqrt{-1}\kappa\wedge\bar{\kappa}, \\
d\kappa &= \alpha\wedge\kappa + \frac{\bar{\mathsf{c}}}{\mathsf{c}}\boldsymbol{K}_5\;\kappa\wedge\zeta + \frac{1}{\bar{\mathsf{c}}}\boldsymbol{K}_6\;\kappa\wedge\bar{\kappa} + \zeta\wedge\bar{\kappa}, \\
d\zeta &= (\alpha-\bar{\alpha})\wedge\zeta + \frac{1}{\mathsf{c}}\boldsymbol{Z}_5\;\kappa\wedge\zeta + \frac{1}{\bar{\mathsf{c}}^2}\boldsymbol{Z}_6\;\kappa\wedge\bar{\kappa} + \frac{1}{\bar{\mathsf{c}}}\boldsymbol{Z}_8\;\zeta\wedge\bar{\kappa} + \frac{\mathsf{c}}{\bar{\mathsf{c}}}\boldsymbol{Z}_9\;\zeta\wedge\bar{\zeta}.
\end{aligned}
$$

Lastly, by introducing the modified Maurer-Cartan 1-form

$$
\alpha =: \boldsymbol{\alpha} - \big(\tfrac{1}{\mathsf{c}}\overline{\boldsymbol{K}}_6 - \tfrac{1}{\mathsf{c}}\boldsymbol{R}_1\big)\kappa - \Big(\tfrac{\bar{\mathsf{c}}}{\mathsf{c}}\tfrac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\Big)\zeta - \big(\tfrac{1}{\bar{\mathsf{c}}}\boldsymbol{B}\big)\bar{\kappa},
$$

the final absorbed equations become:

$$
\begin{aligned}
d\rho &= (\boldsymbol{\alpha}+\overline{\boldsymbol{\alpha}})\wedge\rho + \sqrt{-1}\kappa\wedge\bar{\kappa}, \\
d\kappa &= \boldsymbol{\alpha}\wedge\kappa + \zeta\wedge\bar{\kappa}, \\
d\zeta &= (\boldsymbol{\alpha}-\overline{\boldsymbol{\alpha}})\wedge\zeta + \tfrac{1}{\mathsf{c}}(\boldsymbol{Z}_5 - \overline{\boldsymbol{Z}}_8)\;\kappa\wedge\zeta + \tfrac{1}{\bar{\mathsf{c}}^2}\boldsymbol{Z}_6\;\kappa\wedge\bar{\kappa}.
\end{aligned}
\tag{6.5}
$$

Looking at the expressions of $\boldsymbol{Z}_5, \overline{\boldsymbol{Z}}_8, \boldsymbol{Z}_6$ and comparing with the introduction confirms

$$
\boldsymbol{Z}_5 - \overline{\boldsymbol{Z}}_8 = \boldsymbol{I}_0, \qquad\qquad \boldsymbol{Z}_6 = \boldsymbol{V}_0.
$$

Before we proceed to terminate the $\{e\}$-structure, the differential of any function G of all variables $(z_1, z_2, \overline{z}_1, \overline{z}_2, v, \mathsf{c}, \bar{\mathsf{c}})$ within the full coframe $\big\{\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \boldsymbol{\alpha}, \overline{\boldsymbol{\alpha}}\big\}$

$$
dG = \partial_{\boldsymbol{\alpha}}(G)\;\boldsymbol{\alpha} + \partial_{\overline{\boldsymbol{\alpha}}}(G)\;\overline{\boldsymbol{\alpha}} + \partial_\rho(G)\;\rho + \partial_\kappa(G)\;\kappa + \partial_\zeta(G)\;\zeta + \partial_{\overline{\kappa}}(G)\;\overline{\kappa} + \partial_{\overline{\zeta}}(G)\;\overline{\zeta}.
$$

expresses explicitly in terms of the derivations

$$
\begin{aligned}
\partial_\rho(\cdot) &= \frac{1}{\mathsf{c}\bar{\mathsf{c}}}\mathscr{T}(\cdot), \\
\partial_\kappa(\cdot) &= \frac{1}{\mathsf{c}}\mathscr{L}_1(\cdot) - \frac{1}{\mathsf{c}}\frac{\boldsymbol{B}}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\mathscr{K}(\cdot) + \big(\boldsymbol{R}_1 - \overline{\boldsymbol{K}}_6\big)\,\partial_{\mathsf{c}}(\cdot) - \boldsymbol{B}\,\partial_{\bar{\mathsf{c}}}(\cdot), \\
\partial_\zeta(\cdot) &= \frac{\bar{\mathsf{c}}}{\mathsf{c}}\frac{1}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\,\mathscr{K}(\cdot) - \bar{\mathsf{c}}\frac{\mathscr{L}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\partial_{\mathsf{c}}(\cdot), \\
\partial_{\boldsymbol{\alpha}}(\cdot) &= \mathsf{c}\,\partial_{\mathsf{c}}(\cdot),
\end{aligned}
\tag{6.6}
$$

the unwritten vector fields $\partial_{\overline{\kappa}}$, $\partial_{\overline{\zeta}}$, $\partial_{\overline{\boldsymbol{\alpha}}}$ being complex conjugate of ∂_κ, ∂_ζ, $\partial_{\boldsymbol{\alpha}}$, respectively.

7. Termination of the $\{e\}$-structure: end of proof of Theorem 1.1

In the structure equations finalized above

$$\begin{aligned} d\rho &= (\alpha+\overline{\alpha})\wedge\rho+\sqrt{-1}\kappa\wedge\overline{\kappa}, \\ d\kappa &= \alpha\wedge\kappa+\zeta\wedge\overline{\kappa}, \\ d\zeta &= (\alpha-\overline{\alpha})\wedge\zeta+\tfrac{1}{\mathsf{c}}\boldsymbol{I}_0\,\kappa\wedge\zeta+\tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{V}_0\,\kappa\wedge\overline{\kappa}, \end{aligned}$$

let us introduce (abbreviate)

$$\boldsymbol{I}:=\tfrac{1}{\mathsf{c}}\boldsymbol{I}_0, \qquad \boldsymbol{V}:=\tfrac{1}{\overline{\mathsf{c}}^2}\boldsymbol{V}_0, \qquad \psi:=-\boldsymbol{I}\,\zeta-\boldsymbol{V}\,\overline{\kappa},$$

so that

$$d\zeta=(\boldsymbol{\alpha}-\overline{\boldsymbol{\alpha}})\wedge\zeta+\psi\wedge\kappa.$$

Taking exterior derivatives to exploit $0=d\circ d$, for instance

$$0 \;=\; \big(d\alpha+d\overline{\alpha}\big)\wedge\rho-\big(\alpha+\overline{\alpha}\big)\wedge d\rho+\sqrt{-1}d\kappa\wedge\overline{\kappa}-\sqrt{-1}\kappa\wedge d\overline{\kappa}$$

and replacing $d\rho$, $d\kappa$, $d\zeta$ in the obtained 3 equations, we obtain

$$\tag{7.1} \begin{aligned} 0 &= (d\boldsymbol{\alpha}+d\overline{\boldsymbol{\alpha}})\wedge\rho, \\ 0 &= (d\boldsymbol{\alpha}-\zeta\wedge\bar{\zeta}+\boldsymbol{I}\,\zeta\wedge\bar{\kappa})\wedge\kappa, \\ 0 &= (d\boldsymbol{\alpha}-d\overline{\boldsymbol{\alpha}})\wedge\zeta-(\boldsymbol{\alpha}-\overline{\boldsymbol{\alpha}})\wedge d\zeta+d\psi\wedge\kappa-\psi\wedge\alpha\wedge\kappa. \end{aligned}$$

In the second equation of (7.1), Cartan's lemma provides a 1-form A with

$$\tag{7.2} d\boldsymbol{\alpha}=\zeta\wedge\bar{\zeta}-\boldsymbol{I}\,\zeta\wedge\bar{\kappa}+A\wedge\kappa.$$

Decomposing A along the coframe

$$\tag{7.3} A=A_\rho\rho+A_\kappa\kappa+A_\zeta\zeta+A_{\overline{\kappa}}\overline{\kappa}+A_{\bar{\zeta}}\bar{\zeta}+A_{\boldsymbol{\alpha}}\boldsymbol{\alpha}+A_{\overline{\boldsymbol{\alpha}}}\overline{\boldsymbol{\alpha}},$$

we want to determine these seven coefficients functions. Substituting (7.2) and (7.3) into the first equation of (7.1), we realize that

$$A_\zeta=0, \qquad A_{\bar{\kappa}}\text{ is real}, \qquad A_{\bar{\zeta}}=\overline{\boldsymbol{I}}, \qquad A_{\boldsymbol{\alpha}}=A_{\overline{\boldsymbol{\alpha}}}=0,$$

and so

$$\tag{7.4} d\boldsymbol{\alpha}=\zeta\wedge\bar{\zeta}-\boldsymbol{I}\,\zeta\wedge\bar{\kappa}+A_\rho\rho\wedge\kappa+A_{\bar{\kappa}}\bar{\kappa}\wedge\kappa+\overline{\boldsymbol{I}}\,\bar{\zeta}\wedge\kappa.$$

Next, inserting this $d\boldsymbol{\alpha}$ into the third equation of (7.1), and wedging $(\cdot)\wedge\overline{\zeta}$, we obtain the supplementary information

$$\tag{7.5} A_\rho=0, \qquad 0=2A_{\bar{\kappa}}\,\bar{\kappa}\wedge\kappa\wedge\zeta\wedge\bar{\zeta}+\overline{\boldsymbol{\alpha}}\wedge\psi\wedge\kappa\wedge\bar{\zeta}+d\psi\wedge\kappa\wedge\bar{\zeta}.$$

Now, we expand $d\psi$ so that

$$\begin{aligned} d\psi\wedge\kappa\wedge\bar{\zeta} &= \big(-d\boldsymbol{I}\wedge\zeta-\boldsymbol{I}\,d\zeta-d\boldsymbol{V}\wedge\bar{\kappa}-\boldsymbol{V}\,d\bar{\kappa}\big)\wedge\kappa\wedge\bar{\zeta} \\ &= -\big(\partial_{\overline{\kappa}}(\boldsymbol{I})-\partial_\zeta(\boldsymbol{V})\big)\,\kappa\wedge\bar{\kappa}\wedge\zeta\wedge\bar{\zeta}+\cdots. \end{aligned}$$

By inspecting the coefficient of $\kappa\wedge\overline{\kappa}\wedge\zeta\wedge\overline{\zeta}$ on the right side of equation (7.5), one could think the $\{e\}$-structure would terminate by declaring

$$A_{\bar{\kappa}}:=-\tfrac{1}{2}\big(\partial_{\overline{\kappa}}(\boldsymbol{I})-\partial_\zeta(\boldsymbol{V})\big),$$

which is a secondary invariant. But to make sure that this assignement makes sense, we must still argue that the right-hand side is *real-valued*, and this requires some computation.

Applying (6.6), it comes

$$\partial_{\overline{\kappa}}(I)-\partial_{\zeta}(V)=\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\left(\overline{\mathscr{L}}_1(I_0)-\overline{B}\frac{\overline{\mathscr{K}}(I_0)}{\mathscr{L}(\overline{k})}+BI_0-\frac{\mathscr{K}(V_0)}{\overline{\mathscr{L}}(k)}\right)$$
$$=\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\left(\overline{\mathscr{L}}_1(Z_5)-\overline{\mathscr{L}}_1(\overline{Z}_8)-\overline{B}\frac{\overline{\mathscr{K}}(Z_5)}{\mathscr{L}_1(\overline{k})}+\overline{B}\frac{\overline{\mathscr{K}}(\overline{Z}_8)}{\mathscr{L}_1(\overline{k})}+BZ_5-B\overline{Z}_8-\frac{\mathscr{K}(Z_6)}{\overline{\mathscr{L}}_1(k)}\right).$$

Lemma 7.6. *One has the following identity*

$$\overline{\mathscr{L}}_1(Z_5)-\frac{\mathscr{K}(Z_6)}{\overline{\mathscr{L}}_1(k)}=\overline{B}\frac{\overline{\mathscr{K}}(Z_5)}{\mathscr{L}_1(\overline{k})}+Z_5K_6-Z_6K_5-\mathscr{L}_1(Z_8)+B\frac{\mathscr{K}(Z_8)}{\overline{\mathscr{L}}_1(k)}+Z_8\overline{K}_6+Z_9\overline{Z}_6.$$

Proof. Starting from $d\zeta_0'$ in (6.4), it suffices to capture the coefficient of $\kappa_0\wedge\overline{\kappa}_0\wedge\zeta_0'$ in $0=d\circ d\zeta_0'$, which is, without providing intermediate computations

$$0=\partial_{\overline{\kappa}_0}(Z_5)-Z_5K_6-Z_5Z_8-\partial_{\zeta_0'}(Z_6)+Z_6K_5+\partial_{\kappa_0}(Z_8)+Z_8Z_5-Z_8\overline{K}_6-Z_9\overline{Z}_6.$$

Using (6.2) and reorganizing leads to the result. □

Substituting this identity into $A_{\bar{k}}$ above, we can therefore rewrite

$$A_{\bar{\kappa}}=-\frac{1}{2\mathsf{c}\overline{\mathsf{c}}}\left(Z_9\overline{Z}_6-K_5Z_6+-\mathscr{L}_1(Z_8)-\overline{\mathscr{L}}_1(\overline{Z_8})+B\frac{\mathscr{K}(Z_8)}{\overline{\mathscr{L}}_1(k)}+\overline{B}\frac{\overline{\mathscr{K}}(\overline{Z}_8)}{\mathscr{L}_1(\overline{k})}-Z_8\overline{B}-\overline{Z}_8B\right),$$

and observing lastly that $Z_9=-\overline{K}_5$, we conclude that $A_{\bar{\kappa}}$ is indeed real-valued. Thus the $\{e\}$-structure is finally complete, and we have therefore fully proved Theorem 1.1. □

We can attribute a name to the 'horizontal part' of $A_{\overline{\kappa}}$, namely to the following function defined on M independently of c, $\overline{\mathsf{c}}$

$$\begin{aligned}Q_0&:=\mathsf{c}\overline{\mathsf{c}}\,A_{\overline{\kappa}}\;=\;\overline{\mathsf{c}}\mathsf{c}\,\overline{A_{\overline{\kappa}}}\quad\text{(real-valued)}\\&=-\,\frac{1}{2}\left(\overline{\mathscr{L}}_1\big(I_0\big)-\overline{B}\frac{\overline{\mathscr{K}}\big(I_0\big)}{\mathscr{L}_1(\overline{k})}+BI_0-\frac{\mathscr{K}\big(V_0\big)}{\overline{\mathscr{L}}_1(k)}\right),\end{aligned}\tag{7.7}$$

and realize that its expression can be further normalized thanks to

Proposition 7.8. *One has:*

$$\frac{\overline{\mathscr{K}}(I_0)}{\mathscr{L}_1(\overline{k})}=-\,2\overline{I}_0.$$

Proof. Start from the $\{e\}$-structure

$$\begin{aligned}d\rho&=(\alpha+\overline{\alpha})\wedge\rho+\sqrt{-1}\kappa\wedge\overline{\kappa},\\d\kappa&=\alpha\wedge\kappa+\zeta\wedge\overline{\kappa},\\d\zeta&=(\alpha-\overline{\alpha})\wedge\zeta+\frac{1}{\mathsf{c}}I_0\;\kappa\wedge\zeta+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}V_0\;\kappa\wedge\overline{\kappa},\\d\alpha&=\zeta\wedge\overline{\zeta}-\frac{1}{\mathsf{c}}I_0\;\zeta\wedge\overline{\kappa}+\frac{1}{\mathsf{c}\overline{\mathsf{c}}}Q_0\;\kappa\wedge\overline{\kappa}+\frac{1}{\overline{\mathsf{c}}}\overline{I}_0\;\overline{\zeta}\wedge\kappa,\end{aligned}$$

apply $d(\cdot)$ to the third equation $d\zeta$, use $d\circ d\equiv 0$, and wedge on both sides with $\alpha\wedge\overline{\alpha}\wedge\rho\wedge\overline{\kappa}$, to obtain

$$\begin{aligned}0=\;&d\alpha\wedge\zeta\wedge\alpha\wedge\overline{\alpha}\wedge\rho\wedge\overline{\kappa}-d\overline{\alpha}\wedge\zeta\wedge\alpha\wedge\overline{\alpha}\wedge\rho\wedge\overline{\kappa}\\&+\partial_{\overline{\zeta}}\left(\frac{1}{\mathsf{c}}I_0\right)\overline{\zeta}\wedge\kappa\wedge\zeta\wedge\alpha\wedge\overline{\alpha}\wedge\rho\wedge\overline{\kappa},\end{aligned}$$

where $\partial_{\overline{\zeta}}$ is the following vector field coming from (6.6)

$$\partial_{\overline{\zeta}}=\frac{\mathsf{c}}{\overline{\mathsf{c}}}\frac{1}{\mathscr{L}_1(\overline{k})}\,\overline{\mathscr{K}}-\mathsf{c}\frac{\overline{\mathscr{L}}_1(\overline{k})}{\mathscr{L}_1(\overline{k})}\partial_{\overline{\mathsf{c}}}.$$

Then using $d\alpha$ and $d\overline{\alpha}$ from the $\{e\}$-structure, we obtain the desired identity. $\square$

Thus we recover the expression of $\boldsymbol{Q}_0$ shown in the introduction. More advanced (and nontrivial) computations performed in [4] provide an alternative expression which immediately shows that $\boldsymbol{Q}_0$ is real

$$
\begin{aligned}
\boldsymbol{Q}_0 \;:=\; 2\,\mathsf{Re}\bigg\{ & \frac{1}{9}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(k)\big)\,\overline{\mathscr{L}}_1\big(\mathscr{L}_1(k)\big)^2}{\overline{\mathscr{L}}_1(k)^4} - \\
& - \frac{1}{9}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\big)\,\overline{\mathscr{L}}_1\big(\mathscr{L}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^3} - \frac{1}{9}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(k)\big)\,\overline{\mathscr{L}}_1\big(\mathscr{L}_1(k)\big)\,\overline{P}}{\overline{\mathscr{L}}_1(k)^3} - \\
& - \frac{1}{9}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(k)\big)\,\overline{\mathscr{L}}_1\big(\mathscr{L}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^2} + \frac{1}{9}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\big)\,\overline{P}}{\overline{\mathscr{L}}_1(k)^2} - \\
& - \frac{2}{9}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(k)\big)\,\overline{P}}{\overline{\mathscr{L}}_1(k)} - \frac{1}{9}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\,P}{\overline{\mathscr{L}}_1(k)} + \frac{1}{3}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\big)}{\overline{\mathscr{L}}_1(k)} + \frac{1}{6}\,\overline{\mathscr{L}}_1(P)\bigg\} \\
& - \frac{1}{9}\,|\overline{P}|^2 + \frac{1}{3}\,\bigg|\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)}\bigg|^2.
\end{aligned}
$$

8. Representation by Vector Fields

By a result of Gaussier-Merker [10], the Lie algebra of infinitesimal CR automorphisms of the tube over future light cone M_{LC} is generated by the following 10 holomorphic vector fields

$$
\begin{aligned}
X^1 &= \sqrt{-1}\partial_w, \\
X^2 &= z_1\partial_{z_1} + 2w\partial_w, \\
X^3 &= \sqrt{-1}z_1\partial_{z_1} + 2\sqrt{-1}z_2\partial_{z_2}, \\
X^4 &= (z_2 - 1)\partial_{z_1} - 2z_1\partial_w, \\
X^5 &= (\sqrt{-1} + \sqrt{-1}z_2)\partial_{z_1} - 2\sqrt{-1}z_1\partial_w, \\
X^6 &= z_1z_2\partial_{z_1} + (z_2^2 - 1)\partial_{z_2} - z_1^2\partial_w, \\
X^7 &= \sqrt{-1}z_1z_2\partial_{z_1} + (\sqrt{-1}z_2^2 + \sqrt{-1})\partial_{z_2} - \sqrt{-1}z_1^2\partial_w, \\
X^8 &= \sqrt{-1}wz_1\partial_{z_1} - \sqrt{-1}z_1^2\partial_{z_2} + \sqrt{-1}w^2\partial_w, \\
X^9 &= (z_1^2 - wz_2 - w)\partial_{z_1} + (2z_1z_2 + 2z_1)\partial_{z_2} + 2wz_1\partial_w, \\
X^{10} &= (-\sqrt{-1}z_1^2 + \sqrt{-1}wz_2 - \sqrt{-1}w)\partial_{z_1} + (-2\sqrt{-1}z_1z_2 + 2\sqrt{-1}z_1)\partial_{z_2} - 2\sqrt{-1}wz_1\partial_w.
\end{aligned}
$$

For a $\mathscr{C}^\omega$ rigid hypersurface $M^5 \subset \mathbb{C}^3$, define the Lie pseudogroup

$$\mathsf{Hol}_{\mathsf{rigid}}(M) \;:=\; \Big\{h:\; M \longrightarrow M \text{ local rigid biholomorphism}\Big\}.$$

Its Lie algebra, obtained by differentiating 1-parameter local groups of rigid biholomorphisms, is:

$$
\begin{aligned}
\mathsf{Lie}\big(\mathsf{Hol}_{\mathsf{rigid}}(M)\big) &= \mathfrak{hol}_{\mathsf{rigid}}(M) \\
&:= \bigg\{X = A_1(z_1,z_2)\frac{\partial}{\partial z_1} + A_2(z_1,z_2)\frac{\partial}{\partial z_2} + (\alpha w + B(z_1,z_2))\frac{\partial}{\partial w}: \\
&\qquad (X + \overline{X})|_M \text{ is tangent to } M\bigg\},
\end{aligned}
$$

where A_1, A_2, B are holomorphic functions of only (z_1, z_2), and where $\alpha \in \mathbb{R}$.

Hence, we can easily see that the flows of the vector fields X^i for $1 \leqslant i \leqslant 7$ are rigid, and by argument of Cartan's equivalence method in the previous sections, these are the only ones.

In the interest of comparing with the Maurer-Cartan coframe on M_{LC}, we provide the following table of Lie brackets of the first 7 vector fields

	X^1	X^2	X^3	X^4	X^5	X^6	X^7
X^1	0	$2X^1$	0	0	0	0	0
X^2		0	0	$-X^4$	$-X^5$	0	0
X^3			0	X^5	$-X^4$	$2X^7$	$-2X^6$
X^4				0	$4X^1$	$-X^4$	$-X^5$
X^5					0	X^5	$-X^4$
X^6						0	$-2X^3$
X^7							0

One can deduce directly from the table that the Lie algebra $\mathfrak{hol}_{\mathsf{rigid}}(M_{\mathsf{LC}})$ is not semi-simple. Indeed, the Killing form applied to the first vector field vanishes for any X in the Lie algebra:

$$\mathsf{trace}\big(\mathsf{ad}(X^1)\mathsf{ad}(X)\big) = 0,$$

and the conclusion follows from Cartan's criterion. This shows that parabolic geometry does not apply to our study.

Let

$$\partial_\rho,\ \partial_\kappa,\ \partial_\zeta,\ \partial_{\bar\kappa},\ \partial_{\bar\zeta},\ \partial_{\boldsymbol{\alpha}},\ \partial_{\bar{\boldsymbol{\alpha}}},$$

be the right-invariant vector fields that are respective duals to the Maurer-Cartan 1-forms of the homogeneous model, and let $\mathfrak{g}$ be the Lie algebra generated by these vector fields. In what follows, we will seek a Lie algebra isomorphism

$$\tau : \mathfrak{hol}_{\mathsf{rigid}}(M_{\mathsf{LC}}) \longrightarrow \mathfrak{g}$$

between $\mathfrak{hol}_{\mathsf{rigid}}(M_{\mathsf{LC}})$ and $\mathfrak{g}$.

We recall the following fact which can be found in Olver [34, page 257]. Consider a set of 1-forms $\theta = \{\theta^1, \dots, \theta^m\}$ on a manifold M producing the fundamental structure equations

$$d\theta^i = \sum_{1\leqslant j<k\leqslant m} T^i_{jk}\, \theta^j \wedge \theta^k \qquad {\scriptstyle (i=1,\dots,m)}.$$

If ∂_{θ^i} are the vector fields dual to θ^i, one has the following commutation relations

$$\big[\partial_{\theta^j}, \partial_{\theta^k}\big] = -\sum_{i=1}^m T^i_{jk}\, \partial_{\theta^i} \qquad {\scriptstyle (1\leqslant i<j\leqslant m)}.$$

Following this formula, and if we adopt the order of indices

$$\rho < \kappa < \zeta < \alpha < \bar\kappa < \bar\zeta < \bar\alpha,$$

the Maurer-Cartan structure equations in Theorem 1.2 therefore provide the following commutator table of the vector fields:

	∂_ρ	∂_κ	∂_ζ	$\partial_{\boldsymbol{\alpha}}$	$\partial_{\bar{\kappa}}$	$\partial_{\bar{\zeta}}$	$\partial_{\bar{\boldsymbol{\alpha}}}$
∂_ρ	0	0	0	∂_ρ	0	0	∂_ρ
∂_κ	0	0	0	∂_κ	$-\sqrt{-1}\partial_\rho$	$\partial_{\bar{\kappa}}$	0
∂_ζ	0	0	0	∂_ζ	$-\partial_\kappa$	$-\partial_{\boldsymbol{\alpha}}+\partial_{\bar{\boldsymbol{\alpha}}}$	$-\partial_\zeta$
$\partial_{\boldsymbol{\alpha}}$	$-\partial_\rho$	$-\partial_\kappa$	$-\partial_\zeta$	0	0	$\partial_{\bar{\zeta}}$	0
$\partial_{\bar{\kappa}}$	0	$\sqrt{-1}\partial_\rho$	∂_κ	0	0	0	$\partial_{\bar{\kappa}}$
$\partial_{\bar{\zeta}}$	0	$-\partial_{\bar{\kappa}}$	$-\partial_{\bar{\boldsymbol{\alpha}}}+\partial_{\boldsymbol{\alpha}}$	$-\partial_{\bar{\zeta}}$	0	0	$\partial_{\bar{\zeta}}$
$\partial_{\bar{\boldsymbol{\alpha}}}$	$-\partial_\rho$	0	∂_ζ	0	$-\partial_{\bar{\kappa}}$	$-\partial_{\bar{\zeta}}$	0

Let $W^1, \dots, W^7$ be vector fields defined by

$$\begin{aligned} W^1 &:= -\frac{\sqrt{-1}}{2}\partial_\rho, & W^4 &:= \partial_\kappa - \partial_{\bar{\kappa}}, \\ & & W^5 &:= \partial_\kappa + \partial_{\bar{\kappa}}, \\ W^2 &:= \partial_{\boldsymbol{\alpha}} + \partial_{\bar{\boldsymbol{\alpha}}}, & W^6 &:= \partial_\zeta + \partial_{\bar{\zeta}}, \\ W^3 &:= \partial_\zeta - \partial_{\bar{\zeta}}, & W^7 &:= -\partial_{\boldsymbol{\alpha}} + \partial_{\bar{\boldsymbol{\alpha}}}. \end{aligned}$$

One can see that

$$[W^i, W^j] = [X^i, X^j].$$

The following map concludes the proof of Proposition 1.3

$$\tau \colon \quad X^i \longmapsto \tau(X^i) := W^i \qquad {\scriptstyle (i=1,\dots,7)}.$$

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