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QUASI-PLURISUBHARMONIC ENVELOPES 2: BOUNDS ON MONGE-AMPÈRE VOLUMES

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ABSTRACT. In [GL21a] we have developed a new approach to L^∞-a priori estimates for degenerate complex Monge-Ampère equations, when the reference form is closed. This simplifying assumption was used to ensure the constancy of the volumes of Monge-Ampère measures.

We study here the way these volumes stay away from zero and infinity when the reference form is no longer closed. We establish a transcendental version of the Grauert-Riemenschneider conjecture, partially answering conjectures of Demailly-Păun [DP04] and Boucksom-Demailly-Păun-Peternell [BDPP13].

Our approach relies on a fine use of quasi-plurisubharmonic envelopes. The results obtained here will be used in [GL21b] for solving degenerate complex Monge-Ampère equations on compact Hermitian varieties.

CONTENTS

INTRODUCTION

The study of complex Monge-Ampère equations on compact Hermitian (non Kähler) manifolds has gained considerable interest in the last decade, after Tosatti and Weinkove established an appropriate version of Yau's theorem in [TW10]. The

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smooth Gauduchon-Calabi-Yau conjecture has been further solved by Székelyhidi-Tosatti-Weinkove [STW17], while the pluripotential theory has been partially extended by Dinew, Kołodziej, and Nguyen [DK12, KN15, Din16, KN19].

As in Yau's original proof [Yau78], the method of [TW10] consists in establishing a priori estimates along a continuity path, and the most delicate estimate turns out again to be the a priori L^∞-estimate. The fact that the reference form is not closed introduces several new difficulties: there are many extra terms to handle when using Stokes theorem, and it becomes non trivial to get uniform bounds on the total Monge-Ampère volumes involved in the estimates.

In [GL21a] we have developed a new approach for establishing uniform a priori estimates, restricting to the context of Kähler manifolds for simplicity. While the pluripotential approach consists in measuring the Monge-Ampère capacity of sublevel sets $(\varphi < -t)$, we directly measure the volume of the latter, avoiding delicate integration by parts. Our approach applies in the Hermitian setting, once certain Monge-Ampère volumes are under control. Understanding the behavior of these volumes is the main focus of this article, while [GL21b] is concerned with the resolution of degenerate complex Monge-Ampère equations.

We let X denote a compact complex manifold of complex dimension n, equipped with a Hermitian metric ω_X. The first difficulty we face is to decide whether

$$v_+(\omega_X) := \sup\left\{\int_X (\omega_X + dd^c\varphi)^n \;:\; \varphi \in \mathrm{PSH}(X,\omega_X)\cap L^\infty(X)\right\}$$

is finite. Here $d = \partial + \overline{\partial}$, $d^c = i(\partial - \overline{\partial})$, and $\mathrm{PSH}(X,\omega_X)$ is the set of ω_X-plurisubharmonic functions: these are functions $u : X \to \mathbb{R}\cup\{-\infty\}$ which are locally given as the sum of a smooth and a plurisubharmonic function, and such that $\omega_X + dd^c u \geq 0$ is a positive current. The complex Monge-Ampère measure $(\omega_X + dd^c u)^n$ is well-defined by [BT82].

Building of works of Chiose [Chi16] and Guan-Li [GL10] we provide several results which ensure that the condition $v_+(\omega_X) < +\infty$ is satisfied:

- for any compact complex manifold X of dimension $n \leq 2$;
- for any threefold which admits a pluriclosed metric $dd^c\tilde{\omega}_X = 0$;
- as soon as there exists a metric $\tilde{\omega}_X$ such that $dd^c\tilde{\omega}_X = 0$ and $dd^c\tilde{\omega}_X^2 = 0$;
- as soon as X belongs to the Fujiki class $\mathcal{C}$.

The Fujiki class is the class of compact complex manifolds that are bimeromorphically equivalent to Kähler manifolds.

We also need to bound the Monge-Ampère volumes from below. Given ω a semi-positive form, we introduce several positivity properties:

- we say ω is *non-collapsing* if there is no bounded ω-plurisubharmonic function u such that $(\omega + dd^c u)^n \equiv 0$;
- ω satisfies condition (B) if there exists a constant $B > 0$ such that

$$-B\omega^2 \leq dd^c\omega \leq B\omega^2 \quad \text{and} \quad -B\omega^3 \leq d\omega\wedge d^c\omega \leq B\omega^3;$$

- we say ω is *uniformly non-collapsing* if

$$v_-(\omega) := \inf\left\{\int_X (\omega + dd^c u)^n \;:\; u \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)\right\} > 0.$$

The non-collapsing condition is the minimal positivity condition one should require. We show in Proposition 2.8 that it implies the *domination principle*, a useful extension of the classical maximum principle. We provide a simple example showing that having positive volume $\int_X \omega^n > 0$ does not prevent from being collapsing (see Example 3.5).

After providing a simplified proof of Kołodziej-Nguyen modified comparison principle (see [KN15, Theorem 0.5] and Theorem 1.5), we show that condition (B) implies non-collapsing. The former condition is e.g. satisfied by any form ω which is the pull-back of a Hermitian form on a singular Hermitian variety.

When ω is closed, simple integration by parts reveal that $v_-(\omega) = \int_X \omega^n$ is positive as soon as ω is positive at some point. Bounding from below $v_-(\omega)$ is a much more delicate issue in general. We show in Proposition 3.4 that ω is uniformly non-collapsing if one restricts to ω-psh functions that are uniformly bounded by a fixed constant M:

$$v_M^-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X, \omega) \text{ with } -M \leq u \leq 0 \right\} > 0.$$

For non uniformly bounded functions we show the following:

Theorem A. *The condition $v_+(\omega_X) < +\infty$ is independent of the choice of ω_X; it is moreover invariant under bimeromorphic change of coordinates.*

The condition $v_-(\omega_X) > 0$ is also independent of the choice of ω_X and invariant under bimeromorphic change of coordinates.

In particular these conditions both hold true if X belongs to the Fujiki class.

We are not aware of a single example of a compact complex manifold such that $v_+(\omega_X) = +\infty$ or $v_-(\omega_X) = 0$. This is an important open problem.

The proof of Theorem A relies on a fine use of quasi-plurisubharmonic envelopes. These envelopes have been systematically studied in [GLZ19] in the Kähler framework. Adapting and generalizing [GLZ19] to this Hermitian setting, we prove in Section 2 the following:

Theorem B. *Given a Lebesgue measurable function $h : X \to \mathbb{R}$, we define the ω-plurisubharmonic envelope of h by $P_\omega(h) := (\sup\{u \in \mathrm{PSH}(X, \omega) \ : \ u \leq h\})^*$, where the star means that we take the upper semi-continuous regularization. If h is bounded below, quasi-lower-semi-continuous, and $P_\omega(h) < +\infty$, then*

- *$P_\omega(h)$ is a bounded ω-plurisubharmonic function;*
- *$P_\omega(h) \leq h$ in $X \setminus P$, where P is pluripolar;*
- *$(\omega + dd^c P_\omega(h))^n$ is concentrated on the contact set $\{P_\omega(h) = h\}$.*

An influential conjecture of Grauert-Riemenschneider [GR70] asked whether the existence of a semi-positive holomorphic line bundle $L \to X$ with $c_1(L)^n > 0$ implies that X is Moishezon (i.e. bimeromorphically equivalent to a projective manifold). This conjecture has been solved positively by Siu in [Siu84] (with complements by [Siu85] and Demailly [Dem85]).

Demailly and Păun have proposed a transcendental version of this conjecture (see [DP04, Conjecture 0.8]): given a nef class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ with $\alpha^n > 0$, they conjectured that α should contain a Kähler current, i.e. a positive closed $(1,1)$-current which dominates a Hermitian form. Recall that the Bott-Chern cohomology group $H^{1,1}_{BC}(X, \mathbb{R})$ is the quotient of closed real smooth $(1,1)$-forms, by the image of $\mathcal{C}^\infty(X, \mathbb{R})$ under the dd^c-operator.

This influential conjecture has been further reinforced by Boucksom-Demailly-Păun-Peternell who proposed a weak transcendental form of Demailly's holomorphic Morse inequalities [BDPP13, Conjecture 10.1]. This stronger conjecture has been solved recently by Witt-Nyström when X is projective [WN19].

Building on works of Chiose [Chi13], Xiao [Xiao15] and Popovici [Pop16] we obtain the following answer to the qualitative part of these conjectures:

Theorem C. *Let $\alpha, \beta \in H^{1,1}_{BC}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. The following properties are equivalent:*

1. *$\alpha - \beta$ contains a Kähler current;*
2. *$v_+(\omega_X) < +\infty$;*
3. *X belongs to the Fujiki class.*

A consequence of our analysis is that the conjectures of Demailly-Păun and Boucksom-Demailly-Păun-Peternell can be extended to non closed forms, making sense outside the Fujiki class. Progresses in the theory of complex Monge-Ampère equations on compact hermitian manifolds have indeed shown that it is useful to consider dd^c-perturbations of non closed nef forms. It is therefore natural to try and consider an extension of Theorem C. These are the contents of Theorem 4.6 (when $\beta = 0$) and Theorem 4.15 (when $\beta \neq 0$).

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1. Non collapsing forms

In the whole article we let X denote a compact complex manifold of complex dimension $n \geq 1$, and we fix ω a smooth semi-positive $(1,1)$-form on X.

1.1. Positivity properties.

1.1.1. *Monge-Ampère operators.* A function is quasi-plurisubharmonic (quasi-psh for short) if it is locally given as the sum of a smooth and a psh function.

Given an open set $U \subset X$, quasi-psh functions $\varphi : U \to \mathbb{R} \cup \{-\infty\}$ satisfying $\omega_\varphi := \omega + dd^c\varphi \geq 0$ in the weak sense of currents are called ω-psh functions on U. Constant functions are ω-psh functions since ω is semi-positive. A $\mathcal{C}^2$-smooth function $u \in \mathcal{C}^2(X)$ has bounded Hessian, hence εu is ω-psh on X if $0 < \varepsilon$ is small enough and ω is positive (i.e. Hermitian).

Definition 1.1. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-plurisubharmonic functions which are not identically $-\infty$.

The set $\mathrm{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, for the L^1-topology. We refer the reader to [Dem, GZ, Din16] for basic properties of ω-psh functions.

The complex Monge-Ampère measure $(\omega + dd^c u)^n$ is well-defined for any ω-psh function u which is *bounded*, as follows from Bedford-Taylor theory: if $\beta = dd^c\rho$ is a Kähler form that dominates ω in a local chart, the function u is β-psh hence the positive currents $(\beta + dd^c u)^j$ are well defined for $0 \leq j \leq n$; one thus sets

$$(\omega + dd^c u)^n := \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (\beta + dd^c u)^j \wedge (\beta - \omega)^{n-j}.$$

We refer to [DK12] for an adaptation of [BT82] to the Hermitian context.

The mixed Monge-Ampère measures $(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$ are also well defined for any $0 \leq j \leq n$, and any bounded ω-psh functions u, v. We recall the following classical inequality (see [GL21a, Lemma 1.3]):

Lemma 1.2. *Let φ, ψ be bounded ω-psh functions in $U \subset X$ such that $\varphi \le \psi$. Then*
$$\mathbf{1}_{\{\psi=\varphi\}}(\omega+dd^c\varphi)^j \wedge (\omega+dd^c\psi)^{n-j} \le \mathbf{1}_{\{\psi=\varphi\}}(\omega+dd^c\psi)^n,$$
for all $1 \le j \le n$.

1.1.2. *Condition (B) and non-collapsing.* We always assume in this article that $\int_X \omega^n > 0$. On a few occasions we will need to assume positivity properties that are possibly slightly stronger:

Definition 1.3. We say ω satisfies condition (B) if there exists $B \ge 0$ such that
$$-B\omega^2 \le dd^c\omega \le B\omega^2 \quad \text{and} \quad -B\omega^3 \le d\omega \wedge d^c\omega \le B\omega^3.$$

Here are three different contexts where this condition is satisfied:

- any Hermitian metric $\omega > 0$ satisfies condition (B);
- if $\pi : X \to Y$ is a desingularization of a singular compact complex variety Y and ω_Y is a Hermitian metric, then $\omega = \pi^*\omega_Y$ satisfies condition (B);
- if ω is semi-positive and closed, then it satisfies condition (B).

Combining these one obtains further settings where condition (B) is satisfied.

Definition 1.4. We say ω is non-collapsing if for any bounded ω-psh function, the complex Monge-Ampère measure $(\omega + dd^c u)^n$ has positive mass: $\int_X \omega_u^n > 0$.

We shall see in Corollary 1.6 below that condition (B) implies non-collapsing.

1.2. **Comparison principle.** The comparison principle plays a central role in Kähler pluripotential theory. Its proof breaks down in the Hermitian setting, as it heavily relies on the closedness of the reference form ω through the preservation of Monge-Ampère masses. In that context the following "modified comparison principle" has been established by Kołodziej-Nguyen [KN15, Theorem 0.2]:

Theorem 1.5. *Assume ω satisfies condition (B) and let u, v be bounded ω-psh functions. For $\lambda \in (0,1)$ we set $m_\lambda = \inf_X\{u - (1-\lambda)v\}$. Then*
$$\left(1 - \frac{4B(n-1)^2 s}{\lambda^3}\right)^n \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega^n_{(1-\lambda)v} \le \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega_u^n.$$
for all $0 < s < \frac{\lambda^3}{32B(n-1)^2}$.

The proof by Kołodziej-Nguyen relies on the main result of [DK12], together with extra fine estimates. We propose here a simplified proof.

Proof. Set $\phi := \max(u, (1-\lambda)v + m_\lambda + s)$, $U_{\lambda,s} := \{u < (1-\lambda)v + m_\lambda + s\}$. For $0 \le k \le n$ we set $T_k := \omega_u^k \wedge \omega_\phi^{n-k}$, and $T_l = 0$ if $l < 0$. Set $a = Bs\lambda^{-3}(n-1)^2$. We are going to prove by induction on $k = 0, 1, ..., n-1$ that

$$(1-4a)\int_{U_{\lambda,s}} T_k \le \int_{U_{\lambda,s}} T_{k+1}. \tag{1.1}$$

The conclusion follows since $(\omega_\phi)^n = (\omega_{(1-\lambda)v})^n$ in the plurifine open set $U_{\lambda,s}$.

We first prove (1.1) for $k = 0$. Since $u \le \phi$, Lemma 1.2 ensures that
$$\mathbf{1}_{\{u=\phi\}}\omega_\phi^n \ge \mathbf{1}_{\{u=\phi\}}\omega_u \wedge \omega_\phi^{n-1}.$$

Observing that $U_{\lambda,s} = \{u < \phi\}$ we infer
$$\int_X dd^c(\phi - u) \wedge \omega_\phi^{n-1} = \int_X (\omega_\phi^n - \omega_u \wedge \omega_\phi^{n-1}) \ge \int_{U_{\lambda,s}} \omega_\phi^n - \int_{U_{\lambda,s}} \omega_u \wedge \omega_\phi^{n-1}.$$

A direct computation shows that

$$\begin{aligned} dd^c\omega_\phi^{n-1} &= (n-1)dd^c\omega\wedge\omega_\phi^{n-2}+(n-1)(n-2)d\omega\wedge d^c\omega\wedge\omega_\phi^{n-3} \\ &\le (n-1)B\omega^2\wedge\omega_\phi^{n-2}+(n-1)(n-2)B\omega^3\wedge\omega_\phi^{n-3}, \end{aligned}$$

since ω satisfies condition (B). As $\phi-u\ge 0$, it follows from Stokes theorem that

$$\int_X dd^c(\phi-u)\wedge\omega_\phi^{n-1}\le(n-1)B\left\{\int_X(\phi-u)\omega^2\wedge\omega_\phi^{n-2}+(n-2)\int_X(\phi-u)\omega^3\wedge\omega_\phi^{n-3}\right\}.$$

Observe that

- $\lambda\omega\le\omega_{(1-\lambda)v}$ hence $\omega^j\wedge\omega_\phi^{n-j}\le\lambda^{-j}(\omega_{(1-\lambda)v})^j\wedge\omega_\phi^{n-j}$,
- $(\omega_{(1-\lambda)v})^j\wedge\omega_\phi^{n-j}=\omega_\phi^n$ in the plurifine open set $U_{\lambda,s}$,
- and $0\le\phi-u\le s$ and $\phi-u=0$ on $X\setminus U_{\lambda,s}$,

to conclude that $\int_X(\phi-u)\omega^j\wedge\omega_\phi^{n-j}\le s\lambda^{-j}\int_{U_{\lambda,s}}\omega_\phi^n$, for $j=2,3$, hence

$$\int_{U_{\lambda,s}}\omega_\phi^n-\int_{U_{\lambda,s}}\omega_u\wedge\omega_\phi^{n-1}\le\int_X dd^c(\phi-u)\wedge\omega_\phi^{n-1}\le\frac{Bs(n-1)^2}{\lambda^3}\int_{U_{\lambda,s}}\omega_\phi^n,$$

since $\lambda^{-2}\le\lambda^{-3}$. This yields (1.1) for $k=0$.

We asume now that (1.1) holds for all $j\le k-1$, and we check that it still holds for k. Observe that

$$\begin{aligned} & dd^c\left(\omega_u^k\wedge\omega_\phi^{n-[k+1]}\right) \\ = \;& kdd^c\omega\wedge\omega_u^{k-1}\wedge\omega_\phi^{n-[k+1]}+(n-[k+1])dd^c\omega\wedge\omega_u^k\wedge\omega_\phi^{n-[k+2]} \\ + \;& 2k(n-[k+1])d\omega\wedge d^c\omega\wedge\omega_u^{k-1}\wedge\omega_\phi^{n-[k+2]}+k(k-1)d\omega\wedge d^c\omega\wedge\omega_u^{k-2}\wedge\omega_\phi^{n-[k+1]} \\ + \;& (n-[k+1])[n-(k+2)]d\omega\wedge d^c\omega\wedge\omega_u^k\wedge\omega_\phi^{n-[k+3]}. \end{aligned}$$

The same arguments as above therefore show that

$$\begin{aligned} \int_{U_{\lambda,s}}(T_k-T_{k+1}) &\le \int_X(T_k-T_{k+1})=\int_X(\phi-u)dd^c(\omega_u^k\wedge\omega_\phi^{n-[k+1]}) \\ &\le \frac{Bs}{\lambda^3}\int_{U_{\lambda,s}}\big(k(k-1)T_{k-2}+2k[n-k]T_{k-1}+(n-[k+1])^2T_k\big) \\ &\le a\left(\frac{1}{(1-4a)^2}+\frac{1}{1-4a}+1\right)\int_{U_{\lambda,s}}T_k\le 4a\int_{U_{\lambda,s}}T_k, \end{aligned}$$

where in the third inequality above we have used the induction hypothesis, while the fourth inequality follows from the upper bound $4a<1/8$. From this we obtain (1.1) for k, finishing the proof. $\square$

Corollary 1.6. *If ω satisfies condition (B) then ω is non-collapsing.*

Proof. It follows from Theorem 1.5 that the domination principle holds (see [LPT, Proposition 2.2]). The latter implies in particular that if u,v are ω-psh and bounded, then $e^{-v}(\omega+dd^cv)^n\ge e^{-u}(\omega+dd^cu)^n\Longrightarrow v\le u$ (see [LPT, Proposition 2.3]). There can thus be no bounded ω-psh function u such that $(\omega+dd^cu)^n=0$. Otherwise the previous inequality applied with a constant function $v=A$ yields $u\ge A$ for any A, a contradiction. $\square$

2. Envelopes

We consider here envelopes of ω-psh functions, extending some results of [GLZ19] that have been established for Kähler manifolds.

2.1. Basic properties.

Definition 2.1. A Borel set $E \subset X$ is (locally) pluripolar if it is locally contained in the $-\infty$ locus of some psh function: for each $x \in X$, there exists an open neighborhood U of x and $u \in \mathrm{PSH}(U)$ such that $E \cap U \subset \{u = -\infty\}$.

Definition 2.2. Given a Lebesgue measurable function $h : X \to \mathbb{R}$, we define the ω-psh envelope of h by

$$P_\omega(h) := \left(\sup\{u \in \mathrm{PSH}(X, \omega) \ : \ u \leq h \text{ quasi-everywhere in } X\}\right)^*,$$

where the star means that we take the upper semi-continuous regularization, while quasi-everywhere means outside a locally pluripolar set.

When ω is Hermitian and h is $\mathcal{C}^{1,1}$-smooth, then so is $P_\omega(h)$ (see [Ber19, CZ19, CM20]) and one can show that

$$(\omega + dd^c P_\omega(h))^n = \mathbf{1}_{\{P_\omega(h)=h\}}(\omega + dd^c h)^n. \tag{2.1}$$

For less regular obstacle h we have the following:

Theorem 2.3. *If h is bounded from below, quasi-l.s.c., and $P_\omega(h) < +\infty$, then*

- *$P_\omega(h)$ is a bounded ω-plurisubharmonic function;*
- *$P_\omega(h) \leq h$ in $X \setminus P$, where P is pluripolar;*
- *$(\omega + dd^c P_\omega(h))^n$ is concentrated on the contact set $\{P_\omega(h) = h\}$.*

Recall that a function h is quasi-lower-semicontinuous (quasi-l.s.c.) if for any $\varepsilon > 0$, there exists an open set G of capacity smaller than ε such that h is continuous in $X \setminus G$. Quasi-psh functions are quasi-continuous (see [BT82]), as well as differences of the latter.

Proof. The proof is an adaptation of [GLZ19, Proposition 2.2, Lemma 2.3, Proposition 2.5], which deal with the case when ω is Kähler.

Since $P_\omega(h)$ is bounded from above, up to replacing h with $\min(h, C)$ with $C > \sup_X P_\omega(h)$ we can assume that h is bounded.

Step 1: h is smooth, ω is Hermitian. Building on Berman's work [Ber19], it was shown by Chu-Zhou in [CZ19] that the smooth solutions φ_β to

$$(\omega + dd^c \varphi_\beta)^n = e^{\beta(\varphi_\beta - h)}\omega^n$$

converge uniformly to $P_\omega(h)$ along with uniform $\mathcal{C}^{1,1}$-estimates. As a consequence, the measures $(\omega + dd^c\varphi_\beta)^n$ converge weakly to $(\omega + dd^c P_\omega(h))^n$. For each fixed $\varepsilon > 0$, we have the inclusions of open sets $\{P_\omega(h) < h - 2\varepsilon\} \subset \{\varphi_\beta < h - \varepsilon\}$ for β large enough, yielding

$$\begin{aligned}\int_{\{P_\omega(h)<h-2\varepsilon\}} (\omega + dd^c P_\omega(h))^n &\leq \liminf_{\beta\to+\infty} \int_{\{P_\omega(h)<h-2\varepsilon\}} (\omega + dd^c\varphi_\beta)^n \\ &\leq \liminf_{\beta\to+\infty} \int_{\{P_\omega(h)<h-2\varepsilon\}} e^{-\beta\varepsilon}\omega^n = 0.\end{aligned}$$

Step 2: h is lower semi-continuous, ω is Hermitian. If h is continuous, we can approximate it uniformly by smooth functions h_j. Letting $u_j := P(h_j)$ the previous step ensures that

$$\int_X (h_j - u_j)(\omega + dd^c u_j)^n = 0.$$

As $h_j \to h$ uniformly we also have that $u_j \to u := P(h)$ uniformly and the desired property follows from Bedford-Taylor's convergence theorem.

When h is merely lower semi-continuous, we let (h_j) denote a sequence of continuous functions which increase pointwise to h and set $u_j = P(h_j)$. Then $u_j \nearrow u$ a.e. on X for some bounded function $u \in \mathrm{PSH}(X, \omega)$. Since $u_j \leq h_j \leq h$ quasi-everywhere on X we infer $u \leq h$ quasi-everywhere on X, hence $u \leq P(h)$. For each $k < j$, the second step ensures that

$$\int_{\{u<h_k\}} (\omega + dd^c u_j)^n \leq \int_{\{u_j<h_j\}} (\omega + dd^c u_j)^n = 0.$$

Since $\{u < h_k\}$ is open, letting $j \to +\infty$ and then $k \to +\infty$ we arrive at

$$\int_{\{u<h\}} (\omega + dd^c u)^n = 0.$$

We also have that $P(h) \leq h$ quasi-everywhere on X, hence

$$\int_{\{u<P(h)\}} (\omega + dd^c u)^n = 0,$$

and [LPT, Proposition 2.2] then ensures that $u = P(h)$.

Step 3: h is quasi-l.s.c., ω is Hermitian. By [GLZ19, Lemma 2.4] we can find a decreasing sequence (h_j) of lsc functions such that $h_j \searrow h$ q.e. on X and $h_j \to h$ in capacity. Then $u_j := P(h_j) \searrow u := P(h)$. By Step 2 we know that for all $j > k$,

$$\int_{\{u_k<h\}} (\omega + dd^c u_j)^n \leq \int_{\{u_j<h_j\}} (\omega + dd^c u_j)^n = 0.$$

Since $\{u_k < h\}$ is quasi-open and the functions u_j are uniformly bounded, letting $j \to +\infty$ we obtain

$$\int_{\{u_k<h\}} (\omega + dd^c u)^n = 0.$$

Letting $k \to +\infty$ yields the desired result.

Step 4: the general case. We approximate $\omega \geq 0$ by the Hermitian forms $\omega_j = \omega + j^{-1}\omega_X > 0$. Observe that $j \mapsto u_j = P_{\omega_j}(h)$ decreases to $u = P_\omega(h)$ as j increases to $+\infty$. For $0 < k < j$, the previous step ensures that

$$\int_{\{u_k<h\}} (\omega + j^{-j}\omega_X + dd^c u_j)^n = 0.$$

As the set $\{u_k < h\}$ is quasi-open and u_j is uniformly bounded we can let $j \to +\infty$ and use Bedford-Taylor's convergence theorem to get

$$\int_{\{u_k<h\}} (\omega + dd^c u)^n = 0,$$

We finally let $k \to +\infty$ to conclude. □

For later use we extend the latter result to a setting where $P_\omega(f)$ is not necessarily globally bounded:

Corollary 2.4. *If f is quasi-lower-semicontinuous and $P_\omega(f)$ is locally bounded in a non-empty open set $U \subset X$ then $(\omega + dd^c P_\omega(f))^n$ is a well-defined positive Borel measure in U which vanishes in $U \cap \{P_\omega(f) < f\}$.*

Proof. Let (f_j) be a sequence of l.s.c. functions decreasing to f quasi-everywhere. Then $u_j := P_\omega(f_j)$ is a bounded ω-psh function such that $(\omega + dd^c u_j)^n = 0$ on $\{u_j < f_j\}$. Since u_j decreases to $u := P_\omega(f)$, Bedford-Taylor's convergence theorem ensures that $\omega_{u_j}^n \to \omega_u^n$ in U.

Fix U' a relatively compact open set $U' \Subset U$. For each k fixed the set $\{u_k < f\}$ is quasi open and the functions u_j, u are uniformly bounded in U', hence

$$\liminf_{j\to+\infty} \int_{\{u_k<f\}\cap U'} \omega_{u_j}^n \geq \int_{\{u_k<f\}\cap U'} \omega_u^n,$$

which implies, after letting $k \to +\infty$, that ω_u^n vanishes in $U' \cap \{u < f\}$. We finally let U' increase to U to conclude. □

We shall use later on the following :

Lemma 2.5. *Let u, v be bounded ω-psh functions. Then*

1. $(\omega + dd^c P(\min(u, v)))^n \leq (\omega + dd^c u)^n + (\omega + dd^c v)^n$*;*
2. *if* $(\omega + dd^c u)^n = f dV_X$ *and* $(\omega + dd^c v)^n = g dV_X$*, then*

$$(\omega + dd^c P(\min(u, v)))^n \leq \max(f, g) dV_X,$$

while

$$(\omega + dd^c \max(u, v)))^n \geq \min(f, g) dV_X.$$

Proof. We set $w = P(\min(u, v))$. Since $\min(u, v)$ is quasi-continuous, it follows from Theorem 2.3 that the Monge-Ampère measure ω_w^n has support in

$$\{P(\min(u, v)) = \min(u, v)\} = \{P(\min(u, v)) = u < v\} \cup \{P(\min(u, v)) = v\}.$$

Thus

$$\omega_w^n \leq \mathbf{1}_{\{w=u<v\}} \omega_w^n + \mathbf{1}_{\{w=v\}} \omega_w^n. \tag{2.2}$$

Since $w = P(\min(u, v)) \leq u$ and $w = P(\min(u, v)) \leq u$, Lemma 1.2 yields

$$\mathbf{1}_{\{w=u\}} \omega_w^n \leq \mathbf{1}_{\{w=u\}} \omega_u^n \leq \omega_u^n$$

as well as $\mathbf{1}_{\{w=v\}} \omega_w^n \leq \omega_v^n$. Together with (2.2) we infer $\omega_w^n \leq \omega_u^n + \omega_v^n$ as claimed.

When $(\omega + dd^c u)^n = f dV_X$ and $(\omega + dd^c v)^n = g dV_X$, we obtain

$$\mathbf{1}_{\{w=u<v\}} \omega_w^n \leq \mathbf{1}_{\{w=u<v\}} f dV_X \leq \mathbf{1}_{\{w=u<v\}} \max(f, g) dV_X$$

and $\mathbf{1}_{\{w=v\}} \omega_w^n \leq \mathbf{1}_{\{w=v\}} g dV_X \leq \mathbf{1}_{\{w=u<v\}} \max(f, g) dV_X$, hence

$$\omega_w^n \leq \left\{ \mathbf{1}_{\{w=u<v\}} + \mathbf{1}_{\{w=v\}} \right\} \max(f, g) dV_X \leq \max(f, g) dV_X.$$

The last item follows from

$$(\omega + dd^c \max(\varphi, \psi))^n \geq \mathbf{1}_{\{u\leq v\}} \omega_u^n + \mathbf{1}_{\{v>u\}} \omega_v^n \geq \min(f, g) dV_X.$$

□

2.2. **Locally vs globally pluripolar sets.** A classical result of Josefson asserts that a locally pluripolat set E in $\mathbb{C}^n$ is *globally pluripolar*, i.e. there exists a psh function $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$. This result has been extended to compact Kähler manifolds in [GZ05], and to the Hermitian setting in [Vu19]: if $E \subset X$ is locally pluripolar and ω_X is a Hermitian form, one can find $u \in \mathrm{PSH}(X, \omega_X)$ such that $E \subset \{u = -\infty\}$.

We further extend this result to the case of non-collapsing forms:

Lemma 2.6. *If E is (locally) pluripolar and $\omega \geq 0$ is non-collapsing then $E \subset \{u = -\infty\}$ for some $u \in \mathrm{PSH}(X, \omega)$.*

The proof is a consequence of Theorem 2.3 and analogous results established on Kähler manifolds.

Proof. As in [GZ05, Theorem 5.2] it is enough to check that $V^*_{E,\omega} \equiv +\infty$, where

$$V_{E,\omega}(x) = \sup\{\varphi(x) \ : \ \varphi \in \mathrm{PSH}(X, \omega) \text{ and } \varphi \leq 0 \text{ quasi-everywhere on } E\}.$$

Here quasi-everywhere means outside a locally pluripolar set. If it is not the case then $V^*_{E,\omega}$ is a bounded ω-psh function on X. We can assume that $E \subset U \Subset V \Subset V'$ is contained in a holomorphic chart V'. By Josefson's theorem (see [GZ, Theorem 4.4]) we can find $u \in L^1_{\mathrm{loc}}(V')$ a psh function in V' such that $E \subset \{u = -\infty\}$. Let u_j be a sequence of smooth psh functions in a neighborhood of V such that $u_j \searrow u$. Fix $N \in \mathbb{N}$ and for j large enough we set

$$K_{j,N} := \{x \in V \ : \ u_j(x) \leq -N\}, \ \varphi_{j,N} := V^*_{K_{j,N},\omega},$$

and note that $\varphi_{j,N} \searrow \varphi_N \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ as $j \to +\infty$. We also have that $E \subset \cup_{j \geq 1} K_{j,N}$, hence $0 \leq \varphi_N \leq V^*_{E,\omega}$. We can thus find j_N so large that $\varphi_{j,N} \leq \sup_X V^*_{E,\omega} + 1$ for all $j \geq j_N$.

Let ρ be a smooth psh function in V such that $dd^c\rho \geq \omega$. The Chern-Levine-Nirenberg inequality (see [GZ, Theorem 3.14]) ensures that, for $j \geq j_N$,

$$\begin{aligned}
\int_{K_{j,N}} (\omega + dd^c\varphi_{j,N})^n &\leq \int_{K_{j,N}} (dd^c(\varphi_{j,N} + \rho))^n \\
&\leq \frac{1}{N} \int_V |\varphi_{j,N}| (dd^c(\varphi_{j,N} + \rho))^n \\
&\leq \frac{C}{N},
\end{aligned}$$

for some uniform constant $C > 0$. The function which is zero on $K_{j,N}$ and $+\infty$ elsewhere is lower semi-continuous on X since $K_{j,N}$ is compact. It thus follows from Theorem 2.3 that

$$\int_X (\omega + dd^c\varphi_{j,N})^n = \int_{K_{j,N}} (\omega + dd^c\varphi_{j,N})^n \leq \frac{C'}{N}.$$

Letting $j \to +\infty$ we obtain $\int_X (\omega + dd^c\varphi_N)^n \leq C'/N$. Now $\varphi_N \nearrow \varphi$ as $N \to +\infty$, for some $\varphi \in \mathrm{PSH}(X, \omega)$ which is bounded since $0 \leq \varphi_N \leq V^*_{E,\omega}$. We thus obtain $\int_X (\omega + dd^c\varphi)^n = 0$, yielding a contradiction since ω is non-collapsing and φ is bounded. □

Since locally pluripolar sets are $\mathrm{PSH}(X, \omega)$-pluripolar, arguing as in the proof of [GLZ19, Proposition 2.2], one finally obtains:

Corollary 2.7. *Let f be a Borel function such that $P_\omega(f) \in \mathrm{PSH}(X, \omega)$. Then*

$$P_\omega(f) = \left(\sup\{u \in \mathrm{PSH}(X, \omega) \ : \ u \leq f \text{ in } X\}\right)^*.$$

2.3. **Domination principle.** We now establish the following generalization of the domination principle:

Proposition 2.8. *Assume ω is non-collapsing and fix $c \in [0,1)$. If u, v are bounded ω-psh functions such that $\omega_u^n \leq c\omega_v^n$ on $\{u < v\}$, then $u \geq v$.*

The usual domination principle corresponds to the case $c = 0$ (see [LPT, Proposition 2.2]).

Proof. Fixing $a > 0$ arbitrarily small, we are going to prove that $u \geq v - a$ on X. Assume by contradiction that $E = \{u < v - a\}$ is not empty. Since u, v are quasi-psh, the set E has positive Lebesgue measure. For $b > 1$ we set

$$u_b := P_\omega(bu - (b-1)v).$$

It follows from Theorem 2.3 that $(\omega + dd^c u_b)^n$ is concentrated on the set

$$D := \{u_b = bu - (b-1)v\}.$$

Note also that $b^{-1}u_b + (1 - b^{-1})v \leq u$ with equality on D. Therefore

$$\mathbf{1}_D(\omega + dd^c(b^{-1}u_b + (1 - b^{-1})v))^n \leq \mathbf{1}_D\omega_u^n,$$

as follows from Lemma 1.2, hence

$$\mathbf{1}_D b^{-n}(\omega + dd^c u_b)^n + \mathbf{1}_D(1 - b^{-1})^n(\omega + dd^c v)^n \leq \mathbf{1}_D\omega_u^n.$$

We choose b so large that $(1 - b^{-1})^n > c$. Multiplying the above inequality by $\mathbf{1}_{\{u<v\}}$ and noting that $\omega_u^n \leq c\omega_v^n$ on $\{u < v\}$, we obtain

$$\mathbf{1}_{D\cap\{u<v\}}(\omega + dd^c u_b)^n = 0.$$

Since u_b is bounded and ω is non-collapsing, we know that $\omega_{u_b}^n(D) = \omega_{u_b}^n(X) > 0$. We infer that the set $D \cap \{u \geq v\}$ is not empty, and on this set we have

$$u_b = bu - (b-1)v \geq u \geq -C,$$

since u is bounded. It thus follows that $\sup_X u_b$ is uniformly bounded from below. As $b \to +\infty$ the functions $u_b - \sup_X u_b$ converge to a function u_∞ which is $-\infty$ on E, but not identically $-\infty$ hence it belongs to $\mathrm{PSH}(X, \omega)$. This implies that the set E has Lebesgue measure 0, a contradiction. □

Here is a direct consequence of the domination principle:

Corollary 2.9. *Assume ω is non-collapsing and let u, v be bounded ω-psh functions. Then for all $\varepsilon > 0$,*

$$e^{-\varepsilon v}(\omega + dd^c v)^n \geq e^{-\varepsilon u}(\omega + dd^c u)^n \Longrightarrow v \leq u.$$

Proof. Fix $a > 0$. On the set $\{u < v - a\}$ we have $\omega_u^n \leq e^{-\varepsilon a}\omega_v^n$. Proposition 2.8 thus gives $u \geq v - a$. This is true for all $a > 0$, hence $u \geq v$. □

3. Bounds on Monge-Ampère masses

In the sequel we fix a Hermitian form ω_X on X.

3.1. **Global bounds.** Since the semi-positive $(1,1)$-form ω is not necessarily closed, the mass of the complex Monge-Ampère measures $(\omega + dd^c u)^n$ is (in general) not constantly equal to $V_\omega := \int_X \omega^n > 0$.

Definition 3.1. For $1 \le j \le n$ we consider

$$v_{-,j}(\omega) := \inf\left\{\int_X (\omega + dd^c u)^j \wedge \omega^{n-j},\ u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)\right\}$$

and

$$v_{+,j}(\omega) := \sup\left\{\int_X (\omega + dd^c u)^j \wedge \omega^{n-j},\ u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)\right\}.$$

We set $v_-(\omega) := v_{-,n}(\omega)$ and $v_+(\omega) = v_{+,n}(\omega)$. When $\omega > 0$ is Hermitian, the supremum and infimum in the definition of $v_{+,j}(\omega)$ and $v_{-,j}(\omega)$ can be taken over $\mathrm{PSH}(X,\omega) \cap C^\infty(X)$ as follows from Demailly's approximation [Dem92] and Bedford-Taylor's convergence theorem [BT76, BT82].

It is an interesting open problem to determine when $v_-(\omega_X)$ is positive and/or $v_+(\omega_X)$ is finite. These conditions may depend on the complex structure, but they are independent of the choice of Hermitian metric.

3.1.1. *Monotonicity and invariance properties.*

Proposition 3.2. *Let $0 \le \omega_1 \le \omega_2$ be semi-positive $(1,1)$-forms. Then*

$$v_-(\omega_1) \le v_-(\omega_2) \quad \textit{and} \quad v_+(\omega_1) \le v_+(\omega_2). \tag{3.1}$$

Moreover

1) $v_+(\omega_X) < +\infty \iff v_+(\omega'_X) < +\infty$ for any other Hermitian metric ω'_X.

2) $0 < v_-(\omega_X) \iff 0 < v_-(\omega'_X)$ for any other Hermitian metric ω'_X.

Proof. Since any ω_1-psh function u is also ω_2-psh, we obtain

$$\int_X (\omega_1 + dd^c u)^n \le \int_X (\omega_2 + dd^c u)^n \le v_+(\omega_2).$$

which shows that $v_+(\omega_1) \le v_+(\omega_2)$. We now bound $v_-(\omega_2)$ from below. Let v be a bounded ω_2-psh function and let $u = P_{\omega_1}(v)$ denote its ω_1-psh envelope. Then u is a bounded ω_2-psh function and $u \le v$ on X. Lemma 1.2 and Theorem 2.3 thus ensure that

$$(\omega_1 + dd^c u)^n \le \mathbf{1}_{\{u=v\}}(\omega_2 + dd^c u)^n \le \mathbf{1}_{\{u=v\}}(\omega_2 + dd^c v)^n.$$

We therefore obtain $v_-(\omega_1) \le v_-(\omega_2)$. This proves (3.1).

Let now ω, ω' be two Hermitian metrics (we simplify notations). Observe that $v_\pm(A\omega) = A^n v_\pm(\omega)$ for all $A > 0$. Since $A^{-1}\omega' \le \omega \le A\omega$ for an appropriate choice of the constant $A > 1$, items 1) and 2) follow from (3.1). ☐

We now establish bounds on the mixed Monge-Ampère quantities:

Proposition 3.3.

(1) *One always has $v_{+,1}(\omega) < +\infty$.*
(2) *If ω is Hermitian then $0 < v_{-,1}(\omega)$.*
(3) *If $dd^c\omega^{n-2} = 0$ then $v_{+,2}(\omega) < +\infty$.*
(4) *If $dd^c\omega = 0$ and $dd^c\omega^2 = 0$ then $v_{-,j}(\omega) = v_{+,j}(\omega) = V_\omega \in \mathbb{R}_+^*$.*
(5) *For all $0 \le \ell \le j \le n$ one has $v_{+,\ell}(\omega) \le 2^j v_{+,j}(\omega)$.*
(6) *$v_{+,n-1}(\omega) < +\infty$ if and only if $v_{+,n}(\omega) < +\infty$.*

A Hermitian metric such that $dd^c(\omega^{n-2})=0$ is called Astheno-Kähler. These metrics play an important role in the study of harmonic maps (see [JY93]). A Hermitian metric satisfying $dd^c\omega=0$ is called SKT or pluriclosed in the literature. When $n=3$ the Astheno-Kähler and the pluriclosed condition coincide, and the third item is due to Chiose [Chi16, Question 0.8]. Examples of compact complex manifolds admitting a pluriclosed metric can be found in [FPS04, Ot20].

Condition (4) has been introduced by Guan-Li in [GL10]. It has been shown by Chiose [Chi16] that it is equivalent to the invariance of Monge-Ampère masses: $\int_X(\omega+dd^cu)^n=\int_X\omega^n$ for all smooth ω-psh functions if and only if $dd^c\omega^j=0$ for all $j=1,2$. Note that any compact complex surface admits a Gauduchon metric $dd^c\omega=0$ [Gaud77], which also satisfies $dd^c\omega^2=0$ for bidegree reasons.

Proof. One can assume without loss of generality that $\omega\leq\tilde{\omega}$, where $\tilde{\omega}$ is a Gauduchon metric. It follows that for any $\varphi\in \mathrm{PSH}(X,\omega)$,

$$\int_X(\omega+dd^c\varphi)\wedge\omega^{n-1}\leq\int_X(\omega+dd^c\varphi)\wedge\tilde{\omega}^{n-1}=\int_X\omega\wedge\tilde{\omega}^{n-1},$$

hence $v_{+,1}(\omega)\leq\int_X\omega\wedge\tilde{\omega}^{n-1}<+\infty$.

If ω is Hermitian one can similarly bound from below ω by a Gauduchon form and conclude that $v_{-,1}(\omega)>0$.

We claim that $\int_X(\omega+dd^c\varphi)^2\wedge\omega^{n-2}\leq M$ is uniformly bounded from above when $\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ is normalized and $dd^c\omega^{n-2}=0$. Indeed

$$\begin{aligned}\int_X(\omega+dd^c\varphi)^2\wedge\omega^{n-2} &= \int_X\omega^n+2\int_X\omega^{n-1}\wedge dd^c\varphi+\int_X\omega^{n-2}\wedge(dd^c\varphi)^2\\ &= \int_X\omega^n+2\int_X\varphi dd^c\omega^{n-1}-\int_X\varphi dd^c\omega^{n-2}\wedge dd^c\varphi.\end{aligned}$$

The latter integral vanishes since $dd^c\omega^{n-2}=0$. The second one is uniformly bounded since the functions φ belong to a compact subset of $L^1(X)$. Altogether this shows that $v_{+,2}(\omega)<+\infty$ if $dd^c(\omega^{n-2})=0$.

Since $dd^c(\omega^2)=2d\omega\wedge d^c\omega+2\omega\wedge dd^c\omega$, the Guan-Li condition is equivalent to $dd^c\omega=0$ and $d\omega\wedge d^c\omega=0$. For $u\in\mathrm{PSH}(X,\omega)\cap\mathcal{C}^\infty(X)$ we use the binomial expansion of the Monge-Ampère measure $(\omega+dd^cu)^n$ to obtain

$$\int_X(\omega+dd^cu)^n=\int_X\omega^n+n\int_X\omega^{n-1}\wedge dd^cu+\cdots+n\int_X\omega\wedge(dd^cu)^{n-1}+\int_X(dd^cu)^n.$$

Observe that $dd^c\{du\wedge d^cu\wedge(dd^cu)^{n-2-j}\}=-(dd^cu)^{n-j}$, while $\int_X(dd^cu)^n=0$ by Stokes theorem, hence

$$\begin{aligned}dd^c\left\{\omega^j\wedge du\wedge d^cu\wedge(dd^cu)^{n-2-j}\right\}&=-\omega^j\wedge(dd^cu)^{n-j}\\ &+\ j\omega^{j-1}\wedge dd^c\omega\wedge du\wedge d^cu\wedge(dd^cu)^{n-2-j}\\ &+\ j(j-1)\omega^{j-2}\wedge d\omega\wedge d^c\omega\wedge du\wedge d^cu\wedge(dd^cu)^{n-2-j}.\end{aligned}$$

If $dd^c\omega=0$ and $d\omega\wedge d^c\omega=0$ we infer from Stokes theorem $\int_X\omega^j\wedge(dd^cu)^{n-j}=0$, hence $\int_X(\omega+dd^cu)^n=\int_X\omega^n$ for all $u\in\mathrm{PSH}(X,\omega)\cap\mathcal{C}^\infty(X)$, showing that $v_-(\omega)=v_+(\omega)=V_\omega$ is both finite and positive. Expanding similarly the mixed Monge-Ampère measure $(\omega+dd^cu)^j\wedge\omega^{n-j}$ one obtains 4).

Observe that for any $\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty$ and $0\leq\ell\leq j\leq n$ one has

$$\int_X(\omega+dd^c\varphi)^\ell\wedge\omega^{n-\ell}\leq\int_X(2\omega+dd^c\varphi)^j\wedge\omega^{n-j}\leq 2^jv_{+,j}(\omega). \tag{3.2}$$

In particular $v_{+,n-1}(\omega) \leq 2^n v_{+,n}(\omega)$ hence $v_{+,n}(\omega) < +\infty \Rightarrow v_{+,n-1}(\omega) < +\infty$. We finally show conversely that $v_{+,n-1}(\omega) < +\infty \Rightarrow v_{+,n}(\omega) < +\infty$ by proving

$$v_{+,n}(\omega) \leq 2^{2n-2} v_{+,n-1}(\omega).$$

Observe indeed that

$$\begin{aligned} 0 &= \int_X (\omega + dd^c\varphi - \omega)^n \\ &= \int_X (\omega + dd^c\varphi)^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (\omega + dd^c\varphi)^{n-k} \wedge \omega^k \\ &\geq \int_X (\omega + dd^c\varphi)^n - \sum_{1 \leq 2k+1 \leq n} \binom{n}{2k+1} (\omega + dd^c\varphi)^{n-2k-1} \wedge \omega^{2k+1}. \end{aligned}$$

Using (3.2) we thus get

$$v_{+,n}(\omega) \leq \sum_{1 \leq 2k+1 \leq n} \binom{n}{2k+1} 2^{n-1} v_{+,n-1}(\omega) = 2^{2n-2} v_{+,n-1}(\omega).$$

□

3.1.2. *Uniformly bounded functions.* Restricting to uniformly bounded ω-psh functions, it is natural to consider

$$v_M^-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X,\omega) \text{ with } -M \leq u \leq 0 \right\}$$

where $M \in \mathbb{R}^+$, and

$$v_M^+(\omega) := \sup \left\{ \int_X (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X,\omega) \text{ with } -M \leq u \leq 0 \right\}.$$

These quantities are always under control as we now explain:

Proposition 3.4. *Assume ω is non-collapsing. For any $M \in \mathbb{R}^+$, one has*

$$0 < v_M^-(\omega) \leq v_M^+(\omega) < +\infty.$$

Proof. The finiteness of $v_M^+(\omega)$ follows easily from integration by parts, it is e.g. a simple consequence of [DK12, Theorem 3.5].

In order to show that $v_M^-(\omega)$ is positive we argue by contradiction. Assume there exists $u_j \in \mathrm{PSH}(X,\omega)$ such that $-M \leq u_j \leq 0$ and $\int_X (\omega + dd^c u_j)^n \leq 2^{-j}$. For $j \in \mathbb{N}$ fixed, the sequence

$$k \mapsto v_{j,k} := P_\omega(\min(u_j, u_{j+1}, \ldots, u_{j+k}))$$

decreases towards a ω-psh function w_j such that $-M \leq w_j \leq 0$. It follows therefore from Lemma 2.5 that

$$\int_X (\omega + dd^c w_j)^n = \lim_{k \to +\infty} \int_X (\omega + dd^c v_{j,k})^n \leq \sum_{\ell=0}^{+\infty} \int_X (\omega + dd^c v_{j+\ell})^n \leq 2^{-j+1}.$$

Thus the sequence $j \mapsto w_j$ increases to a bounded ω-psh function w such that $(\omega + dd^c w)^n = 0$, which yields a contradiction. □

Example 3.5. We provide here an example of a semi-positive form ω such that $\int_X \omega^n > 0$ but ω is collapsing, in particular $v_-(\omega) = 0$. Let $X = Y \times Z$ where Y, Z are two compact complex manifolds of dimension $m \geq 1$, $p \geq 1$ respectively, and $\dim X = n = p + m$. Fix a smooth function u on Y such that $\omega_Y + dd^c u < 0$

is negative in a small open set $U \subset Y$. Let $0 \le \rho \le 1$ be a cut-off function on Y supported in U. The smooth $(1,1)$-form ω defined by

$$\omega = \rho \circ \pi_1(\pi_1^*\omega_Y + \pi_2^*\omega_Z).$$

is semipositive on X and satisfies $\omega(y,z) = 0$ for $y \notin U$.

Set now $\phi := P_\omega(u \circ \pi_1)$ and let $\mathcal{C} := \{\phi = u \circ \pi_1\}$ denote the contact set. The Monge-Ampère measure $(\omega + dd^c\phi)^n$ is concentrated on $\mathcal{C}$. Arguing as in [Ber09, Proposition 3.1] one can show that $\mathcal{C} \subset \{x \in X,\ \omega + dd^c u \circ \pi_1(x) \ge 0\}$. Since $\omega + dd^c(u \circ \pi_1) < 0$ is negative in $U \times Z$, it follows that $\mathcal{C} \subset X \setminus (U \times Z)$. Now $\omega = 0$ outside $U \times Z$, hence

$$(\omega + dd^c\phi)^n \le \mathbf{1}_{\mathcal{C}}(dd^c u \circ \pi_1)^n = 0,$$

because $u \circ \pi_1$ depends only on y. It thus follows that $(\omega + dd^c\phi)^n = 0$ on X.

3.2. **Bimeromophic invariance.**

Lemma 3.6. *Let $f : X \to Y$ be a proper holomorphic map between compact complex manifolds of dimension n, equipped with Hermitian forms ω_X, ω_Y. Then*

- $v_+(\omega_X) < +\infty \Longrightarrow v_+(\omega_Y) < +\infty$*;*
- $v_-(\omega_Y) > 0 \Longrightarrow v_-(\omega_X) > 0$ *if f has connected fibers.*

It follows from Zariski's main theorem that f has connected fibers if it is bimeromorphic.

Proof. Up to rescaling, we can assume that $f^*\omega_Y \le \omega_X$. Fix $\varphi \in \mathrm{PSH}(Y, \omega_Y) \cap L^\infty(Y)$. Then $\varphi \circ f \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$ with

$$\int_Y (\omega_Y + dd^c\varphi)^n = \int_X (f^*\omega_Y + dd^c\varphi \circ f)^n \le \int_X (\omega_X + dd^c\varphi \circ f)^n \le v_+(\omega_X),$$

thus $v_+(\omega_Y) \le v_+(\omega_X)$ and the first assertion is proved.

Consider now $\psi \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$ and set $u = P_{f^*\omega_Y}(\psi)$. The function u is $f^*\omega_Y$, hence plurisubharmonic on the fibers of f. If the latter are connected we obtain that u is constant on them, i.e. $u = \varphi \circ f$ for some function $\varphi \in \mathrm{PSH}(Y, \omega_Y) \cap L^\infty(Y)$. Since $(f^*\omega_Y + dd^c u)^n \le \mathbf{1}_{\{u=\psi\}}(f^*\omega_Y + dd^c\psi)^n$, we infer

$$v_-(\omega_Y) \le \int_Y (\omega_Y + dd^c\varphi)^n = \int_X (f^*\omega_Y + dd^c u)^n \le \int_X (\omega_X + dd^c\psi)^n$$

so that $v_-(\omega_Y) \le v_-(\omega_X)$, proving the second assertion. □

We conversely show that the properties $v_+(\omega_X) < +\infty$ and $v_-(\omega_X) > 0$ are invariant under blow ups and blow downs with smooth centers:

Theorem 3.7. *Let X and Y be compact complex manifolds which are bimeromorphically equivalent. Then*

- $v_+(\omega_X) < +\infty$ *if and only if* $v_+(\omega_Y) < +\infty$*;*
- $v_-(\omega_X) > 0$ *if and only if* $v_-(\omega_Y) > 0$.

Proof. A celebrated result of Hironaka ensures that any bimeromorphic map between compact complex manifolds is a finite composition of blow ups and blow downs with smooth centers. We can thus assume that $f : X \to Y$ is the blow up of Y along a smooth center.

We fix ψ a quasi-plurisubharmonic function such that $\pi^*\omega_Y + dd^c\psi \ge \delta\omega_X$. The existence of ψ follows from a classical argument in complex geometry (see [BL70], [FT09, Proposition 3.2]). By Demailly's approximation theorem we can further assume that ψ has analytic singularities. Up to scaling we can assume without loss of generality that $\delta = 1$, and we set $\Omega = \{x \in X \,:\, \psi(x) > -\infty\}$.

We already know by Lemma 3.6 that $v_+(\omega_X) < +\infty \implies v_+(\omega_Y) < +\infty$. Assume conversely that $v_+(\omega_Y) < +\infty$. For any $\varphi \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$,

$$\begin{aligned}\int_X (\omega_X + dd^c\varphi)^n &\leq \int_\Omega (\pi^*\omega_Y + dd^c(\psi+\varphi))^n \\ &\leq \liminf_{j\to+\infty} \int_{\pi(\Omega)} (\pi^*\omega_Y + dd^c(\max[\psi+\varphi, -j]))^n.\end{aligned}$$

The function $u_j = \max[\psi+\varphi, -j]$ is $\pi^*\omega_Y$-psh and bounded in Ω. It is constant on the fibers of π, hence $u_j = v_j \circ \pi$ with $v_j \in \mathrm{PSH}(\pi(\Omega), \omega_Y) \cap L^\infty(\Omega)$. As v_j is bounded, it extends trivially through the analytic set $\pi(\partial\Omega)$ as a bounded ω_Y-psh function. Thus

$$\int_{\pi(\Omega)} (\pi^*\omega_Y + dd^c u_j)^n = \int_Y (\omega_Y + dd^c v_j)^n \leq v_+(\omega_Y)$$

yields $v_+(\omega_X) \leq v_+(\omega_Y) < +\infty$.

We now assume that $v_-(\omega_X) > 0$. Pick $v \in \mathrm{PSH}(Y, \omega_Y) \cap L^\infty(Y)$ and set $u = P_{\omega_X}(v \circ \pi - \psi)$. Observe that $u \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$ and recall that $(\omega_X + dd^c u)^n$ is concentrated on the contact set $\mathcal{C} = \{u + \psi = v \circ \pi\}$ (see Theorem 2.3). Since $u + \psi$ and $v \circ \pi$ are both $\pi^*\omega_Y$-psh, locally bounded in Ω, with $u + \psi \leq v \circ \pi$, it follows from Lemma 1.2 that

$$1_{\mathcal{C}}(\pi^*\omega_Y + dd^c(u+\psi))^n \leq 1_{\mathcal{C}}(\pi^*\omega_Y + dd^c v \circ \pi)^n \leq (\pi^*\omega_Y + dd^c v \circ \pi)^n.$$

Now $\pi^*\omega_Y + dd^c(u+\psi) \geq \omega_X + dd^c u$ and $(\omega_X + dd^c u)^n$ is concentrated on $\mathcal{C}$ so

$$1_{\mathcal{C}}(\pi^*\omega_Y + dd^c(u+\psi))^n \geq (\omega_X + dd^c u)^n.$$

We infer

$$\begin{aligned}v_-(\omega_X) \leq \int_X (\omega_X + dd^c u)^n &\leq \int_{\mathcal{C}} (\pi^*\omega_Y + dd^c(u+\psi))^n \\ &\leq \int_X (\pi^*\omega_Y + dd^c v \circ \pi)^n = \int_Y (\omega_Y + dd^c v)^n,\end{aligned}$$

showing that $v_-(\omega_Y) \geq v_-(\omega_X) > 0$. The reverse implication $v_-(\omega_Y) > 0 \implies v_-(\omega_X) > 0$ follows from Lemma 3.6. □

Recall that a compact complex manifold X belongs to the Fujiki class $\mathcal{C}$ if there exists a holomorphic bimeromophic map $\pi : Y \to X$, where Y is compact Kähler. Since $v_+(\omega_X) = v_-(\omega_X) = \int_X \omega_X^n \in \mathbb{R}_+^*$ when ω_X is a Kähler form, we obtain the following:

Corollary 3.8. *If X belongs to the Fujiki class $\mathcal{C}$ then*

$$0 < v_-(\omega_X) \leq v_+(\omega_X) < +\infty.$$

4. Weak transcendental Morse inequalities

4.1. **Nef and big forms.** Recall that the Bott-Chern cohomology group $H^{1,1}_{BC}(X, \mathbb{R})$ is the quotient of closed real smooth $(1,1)$-forms, by the image of $\mathcal{C}^\infty(X, \mathbb{R})$ under the dd^c-operator. This is a finite dimensional vector space as X is compact.

Nefness and bigness are fundamental positivity properties of holomorphic line bundles in complex algebraic geometry (see [Laz]). Their transcendental counterparts have been defined and studied by Demailly (see [Dem]):

Definition 4.1.

- A cohomology class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is nef if for any $\varepsilon > 0$, one can find a smooth closed real $(1,1)$-form $\theta_\varepsilon \in \alpha$ such that $\theta_\varepsilon \geq -\varepsilon\omega_X$.
- A *Hermitian current* on X is a positive current T of bidegree $(1,1)$ which dominates a Hermitian form, i.e. there exists $\delta > 0$ such that $T \geq \delta\omega_X$.
- A cohomology class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is big if it can be represented by a closed Hermitian current (a Kähler current).

It follows from an approximation result of Demailly [Dem92] that one can weakly approximate a Hermitian current by Hermitian currents with analytic singularities. In particular a big cohomology class can be represented by a Kähler current with analytic singularities.

By analogy with the above setting, we propose the following definitions:

Definition 4.2. Let ω be a smooth real $(1,1)$ form on X.

- We say that ω is nef if for any $\varepsilon > 0$ there exists a smooth quasi-plurisubharmonic function φ_ε such that $\omega + dd^c\varphi_\varepsilon \geq -\varepsilon\omega_X$.
- We say that ω is big if there exists a ω-psh function ρ with analytic singularities such that $\omega + dd^c\rho$ dominates a Hermitian form.

Note that $\mathrm{PSH}(X,\omega)$ is non empty in both cases: indeed $\rho \in \mathrm{PSH}(X,\omega)$ in the latter case, while one can extract $\varphi_{\varepsilon_j} \to \varphi \in \mathrm{PSH}(X,\omega)$ in the former, normalizing the potentials φ_{ε_j} by imposing $\sup_X \varphi_{\varepsilon_j} = 0$.

When X is a compact Kähler manifold and $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is nef with $\alpha^n > 0$, a celebrated result of Demailly-Păun [DP04, Theorem 0.5] ensures the existence of a Kähler current representing α. This result is the key step in establishing a transcendental Nakai-Moishezon criterion (see [DP04, Main theorem]).

We study in the sequel a possible extension of this result to the Hermitian setting. We thus need to extend the definition of v_- to nef forms:

Definition 4.3. If ω is a nef $(1,1)$-form, we set

$$\hat{v}_-(\omega) := \inf_{\varepsilon>0} v_-(\omega + \varepsilon\omega_X).$$

Although the form $\omega + \varepsilon\omega_X$ needs not be semi-positive, one can find by definition a semi-positive form $\omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon$ cohomologous to $\omega + \varepsilon\omega_X$, and it is understood here that $v_-(\omega + \varepsilon\omega_X) := v_-(\omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon)$. By (3.1), the definition of $\hat{v}_-(\omega)$ is independent of the choice of the Hermitian form ω_X.

It is natural to expect that this definition is consistent with the previous one when ω is semi-positive, and that $\hat{v}_-(\omega) = \alpha^n$ when ω is a closed form representing a nef class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$:

Lemma 4.4. *If ω is semi-positive then $\hat{v}_-(\omega) = v_-(\omega)$. If $v_+(\omega_X) < +\infty$ and ω is a closed form representing a nef class in $H^{1,1}_{BC}(X,\mathbb{R})$, then $\hat{v}_-(\omega) = \alpha^n$.*

When X is Kähler, it is classical that any nef class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ satisfies $\alpha^n \geq 0$. This inequality is no longer obvious on an arbitrary hermitian manifold (we thank J.-P.Demailly for emphasizing this issue) but, as a consequence of the above lemma, it remains true when $v_+(\omega_X) < +\infty$.

Proof. Assume first that ω is semi-positive and set $\omega_\varepsilon := \omega + \varepsilon\omega_X$, for $\varepsilon \in (0,1)$. Proposition 3.2 ensures that $v_-(\omega) \leq v_-(\omega_\varepsilon)$, hence $v_-(\omega) \leq \hat{v}_-(\omega)$. On the other hand, for any $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ we have

$$\begin{aligned}\int_X (\omega + dd^c u)^n &= \int_X (\omega_\varepsilon + dd^c u - \varepsilon\omega_X)^n \\ &\geq \int_X (\omega_\varepsilon + dd^c u)^n - C\varepsilon \\ &\geq \hat{v}_-(\omega) - C\varepsilon,\end{aligned}$$

where C is a constant depending on u, but it is harmless as we will let $\varepsilon \to 0$ while keeping u fixed. Doing so we obtain $\int_X (\omega + dd^c u)^n \geq \hat{v}_-(\omega)$, and taking infimum over such u we obtain $v_-(\omega) \geq \hat{v}_-(\omega)$, proving the first statement.

Assume now that ω is closed and $\{\omega\} \in H^{1,1}_{BC}(X,\mathbb{R})$ is nef. We can also assume that $-\omega_X \leq \omega \leq \omega_X$. We pick $\varphi \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X) \cap C^\infty(X)$ and observe that $\mathrm{PSH}(X, \omega + \varepsilon\omega_X) \subset \mathrm{PSH}(X, 2\omega_X)$ for $0 < \varepsilon \leq 1$, hence

$$\int_X (\omega + \varepsilon\omega_X + dd^c\varphi)^n = \int_X (\omega + dd^c\varphi)^n + \sum_{j=1}^n \begin{pmatrix} n \\ j \end{pmatrix} \varepsilon^j \int_X \omega_X^j \wedge (\omega + dd^c\varphi)^{n-j}.$$

Writing $\omega + dd^c\varphi = (2\omega_X + dd^c\varphi) - (2\omega_X - \omega)$, expanding $(\omega + dd^c\varphi)^{n-j}$ accordingly and using $0 \leq 2\omega_X - \omega \leq 3\omega_X$, we obtain that $\left|\int_X \omega_X^j \wedge (\omega + dd^c\varphi)^{n-j}\right|$ is bounded from above by a finite sum of terms $\int_X \omega_X^\ell \wedge (\omega_X + dd^c\varphi)^{n-\ell}$, each of which is bounded from above by $3^n v_+(\omega_X)$. Since $\int_X (\omega + dd^c\varphi)^n = \alpha^n$, we end up with

$$\alpha^n - C\varepsilon v_+(\omega_X) \leq \int_X (\omega + \varepsilon\omega_X + dd^c\varphi)^n \leq \alpha^n + C\varepsilon v_+(\omega_X),$$

using that $\varepsilon^j \leq \varepsilon$ for all $1 \leq j \leq n$. We infer $\hat{v}_-(\omega) = \alpha^n$. □

4.2. **Demailly-Păun conjecture.**

4.2.1. *Hermitian currents.* The following is a natural generalization of [DP04, Conjecture 0.8]:

Question 4.5. Let X be a compact complex manifold. Let ω be a nef $(1,1)$-form such that $\hat{v}_-(\omega) > 0$. Does there exist a ω-psh function φ with analytic singularities such that the current $\omega + dd^c\varphi$ dominates a Hermitian form ?

We provide a partial answer to Question 4.5 following some ideas of Chiose [Chi13]:

Theorem 4.6. *Let ω be a nef $(1,1)$-form.*

- *If $\hat{v}_-(\omega) > 0$ and $v_+(\omega_X) < +\infty$ then ω is big.*
- *Conversely if ω is big and $v_-(\omega_X) > 0$ then $\hat{v}_-(\omega) > 0$.*

Proof. We assume without loss of generality that $\omega \leq \omega_X/2$.

We first assume that $\hat{v}_-(\omega) > 0$, $v_+(\omega_X) < +\infty$, and we prove that ω is big. An application of Hahn-Banach theorem as in [Lam99, Lemma 3.3] shows that the existence of a Hermitian current $\omega + dd^c\psi \geq \delta\omega_X$ is equivalent to the inequalities

$$\int_X \omega \wedge \theta^{n-1} \geq \delta \int_X \omega_X \wedge \theta^{n-1},$$

for all Gauduchon metrics θ. Assume by contradiction that there exists a sequence of Gauduchon metrics θ_j such that

$$\int_X \omega \wedge \theta_j^{n-1} \leq \frac{1}{j} \int_X \omega_X \wedge \theta_j^{n-1}.$$

We can normalize the latter so that $\int_X \omega_X \wedge \theta_j^{n-1} = 1$.

Set $\omega_j = \omega + \frac{1}{j}\omega_X$ and note that $\omega_j \leq \omega_X$ for $j \geq 2$. Since ω is nef, one can find $\psi_j \in \mathcal{C}^\infty(X,\mathbb{R})$ such that $\omega_j + dd^c\psi_j$ is a Hermitian form, hence the main result of [TW10] ensures that there exist constants $C_j > 0$ and $\varphi_j \in \mathrm{PSH}(X,\omega_j) \cap \mathcal{C}^\infty(X)$ such that $\sup_X \varphi_j = 0$ and

$$(\omega_j + dd^c\varphi_j)^n = C_j \omega_X \wedge \theta_j^{n-1}.$$

It follows from Proposition 3.2 that

$$C_j = \int_X (\omega_j + dd^c\varphi_j)^n \geq v_-(\omega_j) \geq \hat{v}_-(\omega) > 0,$$

while by assumption $\int_X (\omega_j + dd^c\varphi_j)^{n-1} \wedge \omega_X \leq M := v_{+,n-1}(\omega_X)$ is bounded from above.

We set $\alpha_j := \omega_j + dd^c\varphi_j$ and consider

$$E := \{x \in X,\ \omega_X \wedge \alpha_j^{n-1} \geq 2M\omega_X \wedge \theta_j^{n-1}\}.$$

This set has small $\omega_X \wedge \theta_j^{n-1}$ measure since

$$\int_E \omega_X \wedge \theta_j^{n-1} \leq \frac{1}{2M} \int_E \omega_X \wedge \alpha_j^{n-1} \leq \frac{1}{2},$$

thus $\int_{X\setminus E} \omega_X \wedge \theta_j^{n-1} \geq \frac{1}{2}$, thanks to the normalization $\int_X \omega_X \wedge \theta_j^{n-1} = 1$.

We can compare ω_X and α_j in $X \setminus E$ since

$$\omega_X \wedge \alpha_j^{n-1} \leq 2M\omega_X \wedge \theta_j^{n-1} = \frac{2M}{C_j}\alpha_j^n \leq \frac{2M}{\hat{v}_-(\omega)}\alpha_j^n.$$

Thus $\alpha_j \geq \frac{\hat{v}_-(\omega)}{2nM}\omega_X$ in $X \setminus E$ and we infer

$$\int_{X\setminus E} \alpha_j \wedge \theta_j^{n-1} \geq \frac{\hat{v}_-(\omega)}{2nM} \int_{X\setminus E} \omega_X \wedge \theta_j^{n-1} \geq \frac{\hat{v}_-(\omega)}{4nM} > 0,$$

which contradicts

$$\begin{aligned} \int_X \alpha_j \wedge \theta_j^{n-1} &= \int_X \omega \wedge \theta_j^{n-1} + \frac{1}{j}\int_X \omega_X \wedge \theta_j^{n-1} + \int_X dd^c\varphi_j \wedge \theta_j^{n-1} \\ &\leq \frac{2}{j}\int_X \omega_X \wedge \theta_j^{n-1} = \frac{2}{j} \to 0, \end{aligned}$$

where $\int_X dd^c\varphi_j \wedge \theta_j^{n-1} = 0$ follows from the Gauduchon property of θ_j.

We next assume that ω is big, $v_-(\omega_X) > 0$, and we prove that $\hat{v}_-(\omega) > 0$ by an argument similar to that of Theorem 3.7. Fix a ω-psh function ψ with analytic singularities such that $\omega + dd^c\psi \geq \delta\omega_X$ for some $\delta > 0$. We can assume that $\delta = 1$ and $\sup_X \psi = 0$. We prove that $v_-(\omega + \varepsilon\omega_X) \geq v_-(\omega_X)$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$, $u \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X) \cap L^\infty(X)$, and set $v = P_{\omega_X}(u - \psi)$. The open set $G = \{\psi > -1\}$ is not empty hence it is non-pluripolar. On G we have $u \leq u - \psi \leq u + 1 \leq \sup_X u + 1$. It follows that v is a bounded ω_X-psh function and $(\omega_X + dd^c v)^n$ is supported on the contact set $\mathcal{C} = \{v = u - \psi\} \subset \{\psi > -\infty\}$. Since $v + \psi \leq u$ with equality on $\{\psi > -\infty\} \cap \mathcal{C}$, Lemma 1.2 ensures that

$$\mathbf{1}_{\{\psi>-\infty\}\cap\mathcal{C}}(\omega + \varepsilon\omega_X + dd^c(v + \psi))^n \leq \mathbf{1}_{\{\psi>-\infty\}\cap\mathcal{C}}(\omega + \varepsilon\omega_X + dd^c u)^n.$$

Using $\omega + dd^c\psi \geq \omega_X$ and the fact that $(\omega_X + dd^c v)^n(\psi = -\infty) = 0$ since v is bounded, we thus arrive at

$$\int_X (\omega_X + dd^c v)^n \leq \int_X (\omega + \varepsilon\omega_X + dd^c u)^n.$$

We thus get $v_-(\omega + \varepsilon\omega_X) \geq v_-(\omega_X) > 0$, for all $\varepsilon > 0$, hence $\hat{v}_-(\omega) > 0$. □

This result provides in particular the following answer to Question 4.5:

Corollary 4.7. *The answer to Question 4.5 is positive if*

- *either $n = 2$ (X is any compact surface);*
- *or $n = 3$ and X admits a pluriclosed metric;*
- *or n is arbitrary and X belongs to the Fujiki class;*
- *orelse n is arbitrary and X admits a Guan-Li metric.*

Let us stress that the 2-dimensional setting is due to Buchdahl [Buch99] and Lamari [Lam99]. The three dimensional case follows from Proposition 3.3.

4.2.2. *Transcendental Grauert-Riemenschneider conjecture.* Let $L \to X$ be a semi-positive holomorphic line bundle with $c_1(L)^n > 0$. An influential conjecture of Grauert-Riemenschneider [GR70] asked whether the existence of such a line bundle implies that X is Moishezon (i.e. bimeromorphically equivalent to a projective manifold).

This conjecture has been solved positively by Siu in [Siu84] (see also [Dem85]). Demailly and Păun have proposed a transcendental version of this conjecture:

Conjecture 4.8. [DP04, Conjecture 0.8] *Let X be a compact complex manifold of dimension n. Assume that X posseses a nef class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ such that $\alpha^n > 0$. Then X belongs to the Fujiki class.*

As a direct consequence of Theorem 4.6, Lemma 4.4, and Corollary 3.8, we obtain the following answer to the transcendental Grauert-Riemenschneider conjecture:

Theorem 4.9. *Let X be a compact n-dimensional complex manifold. Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ be a nef class such that $\alpha^n > 0$. The following are equivalent:*

- *α contains a Kähler current*
- *$v_+(\omega_X) < +\infty$.*

Since a Kähler current with analytic singularities can be desingularized after finitely many blow-ups producing a Kähler form, we obtain:

Corollary 4.10. *Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ be a nef class such that $\alpha^n > 0$. Then X belongs to the Fujiki class if and only if $v_+(\omega_X) < +\infty$.*

4.3. **Transcendental holomorphic Morse inequalities.** The following conjecture has been proposed by Boucksom-Demailly-Păun-Peternell, as a transcendental counterpart to the holomorphic Morse inequalities for integral classes due to Demailly:

Conjecture 4.11. [BDPP13, Conjecture 10.1.ii] *Let X be a compact n-dimensional complex manifold. Let $\alpha, \beta \in H^{1,1}_{BC}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$.*

Then $\alpha - \beta$ contains a Kähler current and $\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$.

Note that this contains [DP04, Conjecture 0.8] as a particular case ($\beta = 0$). This conjecture has recently been established by Witt Nyström [WN19] when X is projective. Building on works of Xiao [Xiao15] and Popovici [Pop16] we propose the following characterization which answers the qualitative part:

Theorem 4.12. *Let $\alpha, \beta \in H^{1,1}_{BC}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. The following are equivalent:*

- *$\alpha - \beta$ contains a Kähler current;*
- *$v_+(\omega_X) < +\infty$.*

Proof. If $\alpha - \beta$ contains a Kähler current, then X belongs to the Fujiki class and we have already observed that $v_+(\omega_X) < +\infty$ (see Corollary 3.8).

We now assume that $v_+(\omega_X) < +\infty$. Let ω and ω' be smooth closed real $(1,1)$-forms representing α and β respectively. We can assume without los of generality that $\omega \leq \frac{\omega_X}{2}$ and $\omega' \leq \frac{\omega_X}{2}$. For each $\varepsilon > 0$ we fix smooth functions $\varphi_\varepsilon \in \mathrm{PSH}(X, \omega + \varepsilon\omega_X)$ and $\psi_\varepsilon \in \mathrm{PSH}(X, \omega' + \varepsilon\omega_X)$ such that $\omega_\varepsilon := \omega + \varepsilon\omega_X + dd^c\varphi_\varepsilon$ and $\omega'_\varepsilon = \omega' + \varepsilon\omega_X + dd^c\psi_\varepsilon$ are hermitian forms.

Assume by contradiction that $\alpha - \beta$ does not contain any Kähler current. It follows from Hahn-Banach theorem as in [Lam99, Lemma 3.3] that there exist Gauduchon metrics η_ε such that

$$\int_X (\omega_\varepsilon - \omega'_\varepsilon) \wedge \eta_\varepsilon^{n-1} \leq \varepsilon \int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1}. \tag{4.1}$$

We normalize η_ε so that $\int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1} = 1$.

Using [TW10] we can find unique constants $c_\varepsilon > 0$ and normalized functions $u_\varepsilon \in \mathrm{PSH}(X, \omega_\varepsilon) \cap \mathcal{C}^\infty(X)$ such that

$$(\omega_\varepsilon + dd^c u_\varepsilon)^n = c_\varepsilon \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1}, \ \sup_X u_\varepsilon = 0.$$

Our normalization for η_ε yields $c_\varepsilon = \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n$. Applying Lemma 4.13 below with $\theta_1 = \omega_\varepsilon + dd^c u_\varepsilon$, $\theta_2 = c_\varepsilon \omega'_\varepsilon$ and $\theta_3 = \eta_\varepsilon$, and recalling that $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$ with $\int_X \theta_1^n = \int_X \theta_2 \wedge \theta_3^{n-1} = c_\varepsilon$, we obtain

$$\left(\int_X (\omega_\varepsilon + dd^c u_\varepsilon) \wedge \eta_\varepsilon^{n-1} \right) \left(\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \right) \geq \frac{c_\varepsilon}{n}.$$

Now $\int_X (\omega_\varepsilon + dd^c u_\varepsilon) \wedge \eta_\varepsilon^{n-1} = \int_X \omega_\varepsilon \wedge \eta_\varepsilon^{n-1}$ because η_ε is a Gauduchon metric, while (4.1) yields $\int_X \omega_\varepsilon \wedge \eta_\varepsilon^{n-1} \leq (1+\varepsilon) \int_X \omega'_\varepsilon \wedge \eta_\varepsilon^{n-1} = (1+\varepsilon)$, hence

$$(1+\varepsilon) \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \geq \frac{1}{n} \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n.$$

We finally claim that, as $\varepsilon \to 0$,

$$\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n \to \alpha^n \text{ and } \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \to \alpha^{n-1} \cdot \beta,$$

which yields the contradiction $n\alpha^{n-1} \cdot \beta \geq \alpha^n$.

We first explain why $\int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n \to \alpha^n$. Stokes theorem yields

$$\begin{aligned} \alpha^n &= \int_X (\omega + dd^c(u_\varepsilon + \varphi_\varepsilon))^n = \int_X (\omega + \varepsilon\omega_X + dd^c(u_\varepsilon + \varphi_\varepsilon) - \varepsilon\omega_X)^n \\ &= \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n + \sum_{j=0}^{n-1} \binom{n}{j} \varepsilon^{n-j} (-1)^{n-j} \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^j \wedge \omega_X^{n-j}. \end{aligned}$$

Since $\omega \leq \frac{\omega_X}{2}$, the function $v_\varepsilon = u_\varepsilon + \varphi_\varepsilon$ is ω_X-psh for $0 < \varepsilon \leq \frac{1}{2}$, hence

$$0 \leq \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^j \wedge \omega_X^{n-j} \leq \int_X (\omega_X + dd^c v_\varepsilon)^j \wedge \omega_X^{n-j} \leq 2^n v_+(\omega_X),$$

as follows from Proposition 3.3. We infer

$$\left| \alpha^n - \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^n \right| \leq \sum_{j=0}^{n-1} \binom{n}{j} \varepsilon^{n-j} 2^n v_+(\omega_X) \leq 4^n \varepsilon\, v_+(\omega_X).$$

The conclusion thus follows by letting $\varepsilon \to 0$.

We similarly can check that

$$\left| \alpha^{n-1} \cdot \beta - \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon \right| \leq 2 \cdot 6^n \varepsilon\, v_+(\omega_X).$$

Using Stokes theorem again we indeed obtain that

$$\begin{aligned}
\alpha^{n-1} \cdot \beta &= \int_X (\omega + dd^c \varphi_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge (\omega' + dd^c \psi_\varepsilon) \\
&= \int_X (\omega_\varepsilon + dd^c u_\varepsilon - \varepsilon \omega_X)^{n-1} \wedge (\omega'_\varepsilon - \varepsilon \omega_X) \\
&= \int_X (\omega_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega'_\varepsilon + O(\varepsilon).
\end{aligned}$$

Each term $\int_X (\omega_X + dd^c v_\varepsilon)^\ell \wedge (\omega_X + dd^c \psi_\varepsilon)^p \wedge \omega_X^q$, with $\ell + p + q = n$, is bounded from above by $3^n v_+(\omega_X)$, as one can check by observing that the function $\frac{v_\varepsilon + \psi_\varepsilon}{3}$ is ω_X-psh with

$$\int_X (\omega_X + dd^c v_\varepsilon)^\ell \wedge (\omega_X + dd^c \psi_\varepsilon)^p \wedge \omega_X^q \leq 3^n \int_X \left(\omega_X + dd^c \frac{v_\varepsilon + \psi_\varepsilon}{3} \right)^n.$$

□

We have used in the previous proof the following observation of Popovici:

Lemma 4.13. *Let $\theta_1, \theta_2, \theta_3$ be hermitian forms on X. Then*

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1} \right) \left(\int_X \theta_1^{n-1} \wedge \theta_2 \right) \geq \frac{1}{n} \left(\int_X \sqrt{\frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}} \theta_1^n \right)^2.$$

In particular if $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$, then

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1} \right) \left(\int_X \theta_1^{n-1} \wedge \theta_2 \right) \geq \frac{1}{n} \left(\int_X \theta_1^n \right)^2.$$

We provide the proof as a courtesy to the reader.

Proof. It follows from Cauchy-Schwarz inequality that

$$\left(\int_X \theta_1 \wedge \theta_3^{n-1} \right) \left(\int_X \theta_1^{n-1} \wedge \theta_2 \right) \geq \left(\int_X \sqrt{\frac{\theta_1 \wedge \theta_3^{n-1}}{\theta_1^n} \frac{\theta_1^{n-1} \wedge \theta_2}{\theta_1^n}} \theta_1^n \right)^2.$$

The elementary pointwise estimate

$$Tr_{\theta_3}(\theta_1) Tr_{\theta_1}(\theta_2) \geq Tr_{\theta_3}(\theta_2).$$

is [Pop16, Lemma 3.1]. Multiplying by $\frac{\theta_3^n}{\theta_1^n}$ it can be reformulated as

$$\frac{\theta_1 \wedge \theta_3^{n-1}}{\theta_1^n} \cdot \frac{\theta_2 \wedge \theta_1^{n-1}}{\theta_1^n} \geq \frac{1}{n} \frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}. \tag{4.2}$$

The first inequality follows. Moreover when $\theta_1^n = \theta_2 \wedge \theta_3^{n-1}$, we infer

$$\int_X \sqrt{\frac{\theta_2 \wedge \theta_3^{n-1}}{\theta_1^n}} \theta_1^n = \int_X \theta_1^n.$$

□

Motivated by possible extensions of the conjectures of Demailly-Păun and Boucksom-Demailly-Păun-Peternell, we introduce the following:

Definition 4.14. Given $\omega_1, \ldots, \omega_n$ hermitian forms we consider

$$v_-(\omega_1, \ldots, \omega_n) := \inf \left\{ \int_X (\omega_1 + dd^c \varphi_1) \wedge \cdots \wedge (\omega_n + dd^c \varphi_n),\ \varphi_j \in \mathcal{P}(\omega_j) \right\},$$

and

$$v_+(\omega_1, \ldots, \omega_n) := \sup \left\{ \int_X (\omega_1 + dd^c \varphi_1) \wedge \cdots \wedge (\omega_n + dd^c \varphi_n),\ \varphi_j \in \mathcal{P}(\omega_j) \right\},$$

where $\mathcal{P}(\omega_j) := \mathrm{PSH}(X, \omega_j) \cap L^\infty(X)$. If the ω_j's are merely nef we set

$$\hat{v}_-(\omega_1, \ldots, \omega_n) := \inf_{\varepsilon > 0} v_-(\omega_1 + \varepsilon \omega_X, \ldots, \omega_n + \varepsilon \omega_X).$$

and

$$\hat{v}_+(\omega_1, \ldots, \omega_n) := \inf_{\varepsilon > 0} v_+(\omega_1 + \varepsilon \omega_X, \ldots, \omega_n + \varepsilon \omega_X).$$

A straightforward generalization of Theorem 4.12 along the lines of Theorem 4.6 is the following:

Theorem 4.15. *Let X be a compact n-dimensional complex manifold such that $v_+(\omega_X) < +\infty$. Let ω, ω' be nef $(1,1)$-forms. If $\hat{v}_-(\omega) > n \hat{v}_+(\omega, \ldots, \omega, \omega')$ then the form $\omega - \omega'$ is big.*

We leave the technical details to the reader.

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