THE KODAIRA DIMENSION OF A GRAVITATIONAL MONOPOLE

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Abstract. This article is a step forward to prove the existence of the solution of the Gravitational monopole equations. We show using simple arguments that surfaces of Kähler type of Kodaira dimension ≥ 0 can be a solution of the Gravitational monopole equation, in particular the complex elliptic surfaces of Kodaira dimension 0 or 1 are the possible candidates.

1. Introduction

Gravitational monopole equations were first introduced in [cf. 1]. If we assume the solutions to the equation exists, the next problem is to understand the nature of the solution and classify them. In this article we propose a classification scheme based on the *Kodaira dimensions* of the Gravitational monopole. We also place a step forward in proving what manifolds come as a solution the Gravitational monopole equation. Some Kähler type surfaces with Kodaira dimension ≥ 0 can be the candidates, in particular some elliptic surfaces with Kodaira dimension 0 or 1 are under investigation. We shall show how simple arguments can be used to reach the conclusions stated above.

2. The Theorems

For the sake of completeness, we provide the definition of Kodaira dimensions of a complex surface. If Σ is of complex dimension m, then the canonical line bundle K_Σ is defined over Σ such that $K_\Sigma = \wedge^{m,0}$, so that the holomorphic sections are holomorphic m-forms on Σ. The Kodaira dimension of Σ is defined in the following way:

$$\text{Kod } (\Sigma) = \limsup_{l\to\infty} \frac{\log\ h^0(\Sigma, K^{\otimes l})}{\log\ l}. \tag{2.1}$$

It is shown by Shafarevich et al in [cf.8] that the above definition coincide with the maximal complex dimension of the image of Σ under pluri-canonical maps to complex projective space, therefore,

$$\text{Kod}(\Sigma) \in \{-\infty, 0, 1, \cdots, m\}. \tag{2.2}$$

Definition 2.1. [Manifold of General type] A compact complex m dimensional manifold is said to be of general type if $\text{Kod}(M) = m$.

It is shown by the author in [cf. 2] that the Yamabe invariant $Y(M) \leq 0$, namely,

Theorem 2.2. *Let M be a Gravitational monopole, then $Y(M) \leq 0$.*

On the other hand Lebrun showed [cf.4] that

Theorem 2.3. *Let M be the underlying 4-manifold of a compact complex surface (M^4, J) with $b_1(M)$ even. Then*

(1). *$Y(M) < 0$ if and only if Kod$(M) = 2$;*
(2). *$Y(M) = 0$ if and only if Kod$(M) = 0$ or 1;*
(3). *$Y(M) > 0$ if and only if Kod$(M) = -\infty$;*

Therefore with the help of theorem (2.3) and theorem (2.2), we deduce the following theorem for a Gravitational manifold,

Theorem 2.4. *Let M be a Gravitational monopole, with $b_1(M) \equiv 0\ mod(2)$, then Kod$(M) \in \{0, 1, 2\}$.*

Proof. Using theorem (2.3) and theorem (2.2) we are reduced to the cases (1) and (2) of theorem (2.3). Hence the conclusion of the theorem. □

Remark 2.5. The motivation behind the theorem (2.4) can be traced back to the fundamental classification problem for compact Riemannian surfaces and classification of Holomorphic vector bundles over them via Narasimhan-Seshadri theorem. One actually gets a one-one correspondence between $\mathrm{Kod}(\Sigma) \in \{-\infty, 0, 1\}$ and $\{Y(\Sigma) > 0, Y(\Sigma) = 0, Y(\Sigma) < 0\}$.

Remark 2.6. Yamabe invariant of the underlying Riemannian manifold of Σ, namely $Y(M)$ can not distinguish between Kodaira dimension 0 and 1.

The following theorem, that actually says, for nonnegative Kodaira dimension of M, the surface does not admit metrics of positive scalar curvature, had been proved by Witten [cf.3] for $b^+(M) > 1$, for $b^+(M) = 1$ in the case of minimal surfaces by LeBrun [cf.5], for non-minimal case by Friedman and Morgan [cf.6].

Theorem 2.7. *Let M be the underlying 4-manifold of a complex surface Σ of Kähler type with Kodaira dimension ≥ 0. Then $Y(M) \leq 0$.*

LeBrun in [cf.7] also proved,

Theorem 2.8. *Let M be the underlying 4-manifold of a complex surface Σ of Kähler type with Kodaira dimension 2, then $Y(M) < 0$. Moreover, if X is the minimal model of M, then*

$$Y(M) = Y(X) = -4\pi\sqrt{2c_1^2(X)}. \tag{2.3}$$

The minimal model X of a surface M of general type is unique, and $c_1^2(X) > 0$.

The following is shown in [cf.4] by LeBrun.

Proposition 1. *Let Σ be a complex elliptic surface and M is the underlying Riemannian four-manifold, then one has the Yamabe invariant of M, $Y(M) \geq 0$.*

3. Towards the solution of Gravitational Monopole Equations.

Using theorems (2.7), (2.8), and Proposition (1) and by theorem (2.4) surfaces of Kähler type of Kodaira dimension ≥ 0 (hence $Y(M) \leq 0$) are possible candidate to the solution of the Gravitational monopole equation. In particular an elliptic surface Σ can be a solution for the Gravitational monopole equation only when the corresponding Riemanninan four manifold M has Yamabe invariant zero, that is $Y(M) = 0$. This gives two options for Kodaira dimension of Σ, namely $\mathrm{Kod}(\Sigma)$ is either 0 or 1. So, one narrows down oneself to the study of surfaces of Kodaira dimension 0 or 1 as a solution of the Gravitational monopole equations. We shall proceed more specifically in this direction in a next paper and will show which elliptic surfaces can be considered as the solutions of the Gravitational monopole equations. Hence the problem of existence of solutions of the Gravitational monopole equations will be resolved.

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